CGL & BL 2026

Proceedings of the Czech Gathering of Logicians & Beauty of Logic

February 4 – 6, 2026

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Organized by

Institute of Computer Science of the Czech Academy of Sciences

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Preface

It is our pleasure to welcome you to the **Czech Gathering of Logicians & Beauty of Logic 2026**, held at the Czech Academy of Sciences from February 4 to 6, 2026.

This year's event is a combination of the Czech Gathering of Logicians with the occasional "Beauty of Logic" conference series. The latter is dedicated to Professor Petr Hájek's research topics, including set theory, arithmetic, fuzzy logic, and logic in data analysis. The programme features six invited lectures and eighteen contributed talks.

We would like to thank the Institute of Computer Science and the Institute of Philosophy of the Czech Academy of Sciences for their support.

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Prague, February 2026

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Conference Program

Wednesday – February 4, 2026	
09:30 – 10:00	*Registration*
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11:00 – 11:30	*Coffee*
11:30 – 12:00	**Han Gao, Daniil Kozhemiachenko, Nicola Olivetti** Paraconsistent Constructive Modal Logic
12:00 – 12:30	**Giuliano Rosella** Modal Weak Kleene Logics Through Variables Inclusion
12:30 – 13:00	**Fabio De Martin Polo** Termination, Countermodels, and Complexity in Bilateral Labeled Sequent Calculi
13:00 – 14:30	*Lunch break*
14:30 – 15:30	**Adam Přenosil (Invited)** Deciding Equations in Conuclear Residuated Lattices
15:30 – 16:00	*Coffee*
16:00 – 16:30	**Marta Bílková, Peter Jipsen** On the Structure of Residuated Po-semigroups as Models of Lambek Calculus and MLL
16:30 – 17:00	**Wesley Fussner, Simon Santschi** Deductive Interpolation in Hájek's Basic Fuzzy Logic
17:00 – 17:30	**Vilém Novák, Petra Murinová** Intermediate Quantifiers and their Syllogisms in Fuzzy Natural Logic
17:30 – 18:00	**Krysztof Krawczyk, Wesley Fussner** Interpolation Properties Among Arbitrary Extensions of RM
19:00	**Conference Dinner** *Institute of Computer Science, Prague 8*
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11:30 – 12:00	**Guillermo Badia, Gaia Petreni, Carles Noguera, Val Tannen** Containment of Conjunctive Queries with Negated Atoms for Databases over Semirings
12:00 – 12:30	**Štěpán Holub, Zuzana Haniková** Formalizations of Set Theory Fragments in Isabelle/HOL
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16:00 – 16:30	**Jiří Raclavský** Inexpressible Propositions and Limits of Knowledge, Belief and Truth
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12:00 – 12:30	**Karel Šebela** When 'Every S is P' Became Hypothetical: Rediscovering Herbart
12:30 – 14:00	*Lunch break*
14:00 – 15:00	**Gustav Šír (Invited)** Neuro-Symbolic Learning With Relational Logic via Differentiable Semantics
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Invited Talks

On Fuzzy Logic, Probability Theory, and Their Cooperation

Tommaso Flaminio
IIIA-CSIC

Since its inception, fuzzy logic has been frequently compared with (and sometimes equated to) probability theory. Over the years, several attempts have been made to clarify the philosophical, mathematical, logical, and applicative differences between these two theories. Interestingly, two crucial papers addressing this topic were both published in 1995:

(1) "Probability Theory and Fuzzy Logic Are Complementary Rather Than Competitive", by Lotfi Zadeh, (see [13]), and

(2) "Probability and Fuzzy Logic", by Petr Hájek, Francesc Esteva, and Lluis Godo [7].

The present talk aims to draw on recent literature that builds upon the paper by Hájek, Esteva, and Godo (2) in order to offer an additional perspective on the discussion initiated by Zadeh's paper (1). Our goal, therefore, is not to further clarify the distinction between fuzzy logic and probability (or uncertainty in general). Rather, we aim to support the idea that these two theories often operate in a cooperative manner.

We will support our general claim by showing how fuzzy logic aids probability theory in its generalization providing a suitable setting for extension of the classical theory of probability in several ways, and how probability theory can be (partially) reduced, interpreted, and represented within a formal fuzzy-logical framework.

Concerning the first we will recall and review the main proposals for generalizing probability theory to fuzzy events as proposed and studied by several scholars within the additive (cf. [9, 10, 11, 12]) and non-additive (cf. [2, 3]) setting.

As for the second, the focus will be on the effect of translating probability formulas to Łukasiewicz language as proposed by Hájek, Esteva, and Godo and the further developed by others ([1, 4, 5] to quote a few).

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Meaningful Formalism and Infinitary Objects

Ansten Klev
Institute of Philosophy, Czech Academy of Sciences

Infinitary objects, such as infinite ordinals, might seem to pose a problem for constructivism. From the basic constructivist tenet that every existential assertion must be backed up by a witness it follows that every object reasoned about must be finitely presentable. Taking an intensional view of mathematical ontology, I shall regard the linguistic presentation of an object as its form. If an object is uniquely determined by its form, as in traditional formalism, then constructivism indeed rules out infinitary objects. If, however, we allow objects to have both form and content, as in the view I call meaningful formalism, then infinitary objects are constructively acceptable. I will explain how the analysis of the content of an object takes the form of a tree. The tree is, in general, infinite, but since it portrays a process, the infinity in question is potential.

Extending Codd's Theorem to Databases Over Semirings

Carles Noguera
University of Siena

Codd's Theorem, a fundamental result of database theory, asserts that relational algebra and relational calculus have the same expressive power on relational databases. In this talk, we will explore Codd's Theorem for databases over semirings and establish two different versions of this result for such databases: the first version involves the five basic operations of relational algebra, while in the second version the division operation is added to the five basic operations of relational algebra. In both versions, the difference operation of relations is given semantics using semirings with monus, while on the side of relational calculus a limited form of negation is used. The reason for considering these two different versions of Codd's theorem is that, unlike the case of ordinary relational databases, the division operation need not be expressible in terms of the five basic operations of relational algebra for databases over an arbitrary positive semiring; in fact, we will show that this inexpressibility result holds for bag databases, as well as for databases over the tropical semiring.

Deciding Equations in Conuclear Residuated Lattices

Adam Přenosil
Institute of Computer Science, Czech Academy of Sciences

Residuated lattices form a wide umbrella class of algebras which allows us to treat from a uniform perspective various superficially quite divergent classes of algebras with implication-like or division-like operations, such as Heyting algebras and lattice-ordered groups. In addition to the familiar constructions of universal algebra (homomorphic images, subalgebras, and so on), the structure theory of residuated lattices prominently features two further constructions, namely so-called nuclear and conuclear images. We will focus on the latter.

A *conucleus* is an interior operator on a residuated lattice whose image is a submonoid. Crucially, the image of the conucleus can also be equipped with the structure of a residuated lattice, albeit one which is not a subalgebra of the original. This allows us to represent residuated lattices of some general kind (Heyting algebras, cancellative commutative residuated lattices) as sitting inside residuated lattices of a much more special kind (Boolean algebras, Abelian lattice-ordered groups) and leverage our deeper understanding of this more special class.

Besides their instrumental role in the theory of residuated lattices, conuclei also have a logical importance of their own, being generalizations of S4 modal box operators on Boolean algebras. (S4 is the modal logic of preorders, as well as the logic of the topological interior operator.) Studying *conuclear residuated lattices*, i.e. residuated lattices equipped with a conucleus, thus amounts to studying substructural variants of the classical modal logic S4.

In this talk, we will consider the problem of deciding the validity of quasi-equations, or equivalently universal sentences, in varieties of integral conuclear residuated lattices. We sketch the tools needed to show that, among others varieties, conuclear MV-algebras and conuclear Abelian ℓ-group cones have decidable quasi-equational theories. In logical terms, this means that a particular version of S4 Łukasiewicz modal logic has a decidable deducibility problem.

As a corollary, this settles positively the long-standing open problem of whether the quasi-equational theory of integral cancellative commutative residuated lattices is decidable, which has not seen much progress since Horčík settled the totally ordered case in 2006. The big open problem remains the decidability of the (quasi-)equational theory of conuclear Abelian ℓ-groups.

Gender Dimension of Disinformers' Confrontational Style

Iva Svačinová
University of Hradec Králové

Petra Vodová
University of Hradec Králové

The rise of illiberalism and strategic political polarization has positioned disinformation as a critical challenge to democratic resilience in Central Europe. While previous research has extensively mapped the thematic content of disinformation—such as anti-Western or anti-vaccination narratives—this study addresses a significant research gap by analyzing the tactical use of gender as a confrontational tool. Focusing on the Czech media ecosystem, the study investigates the argumentative style of disinformation producers, specifically through the use of gendered *ad hominem* attacks on prominent disinformation websites. These platforms represent an illiberal, nationalist, and socially conservative ideological spectrum that actively opposes the consensual "official story" presented by mainstream media and political leaders.

Theoretically, the study adopts a pragma-dialectical approach to argumentation, specifically utilizing van Eemeren's concept of argumentative style. We focus on the *ad hominem* fallacy as a form of "derailed strategic maneuvering" that violates the Freedom Rule by silencing opponents and disqualifying them as serious discussion partners. Drawing on Peng's framework, we distinguish between a compromising confrontational style—used to define differences of opinion in a way that allows for modification—and an uncompromising style, which seeks to exclude differences from resolution entirely.

A central contribution of this research is the introduction and definition of gendered versions of these attacks. While drawing on the concept of *ad feminam*—attacks against women grounded in gender stereotypes—the study also proposes the term *ad masculum* to describe attacks targeting men through emasculation or the questioning of their masculinity. We hypothesize that gendered attacks serve different strategic purposes depending on the audience. For the "primary audience" of disinformers' readers and supporters, third-person *ad hominem* attacks function as cultural markers that build a negative ethos for opponents and reinforce a sense of group belonging. For the "secondary audience" of political opponents and journalists, second-person attacks are designed to induce self-doubt, fear, or exhaustion, effectively chasing them out of the dialectical space.

Our study explores how gendered *ad hominem* attacks are utilized on Czech disinformation websites. Specifically, we examine whether gendered attacks are more prevalent than those drawing from other *topoi*, and whether such attacks target women more frequently than men. Furthermore, we investigate which specific character traits are targeted within these attacks and how these attacks are rhetorically constructed. The methodology employs qualitative content analysis using multiple coding cycles—including structural, descriptive, and inductive coding—to identify specific themes and linguistic choices.

Neuro-Symbolic Learning With Relational Logic via Differentiable Semantics

Gustav Šír
Faculty of Electrical Engineering, Czech Technical University

Modern deep learning has achieved remarkable success in function approximation over vectorial and grid-like data representations. However, it struggles to natively accommodate the rich relational model structures typical of formal logic and ubiquitous in real-world domains, such as relational databases and knowledge-rich data. The emerging field of neuro-symbolic AI aims to address this gap by combining the efficient learnability of neural networks with the expressive power and reasoning capabilities of symbolic logic. In this talk, I will review one such neuro-symbolic research trajectory toward learning from complex relational data and knowledge, approached through the lens of differentiable logic programming. I begin by briefly outlining the limitations of standard neural architectures for relational learning and contrasting them with the representational strengths of classical symbolic approaches rooted in logical data analysis, particularly Inductive Logic Programming (ILP). Viewed through a logical perspective, I then examine Graph Neural Networks (GNNs) as a modern deep learning alternative with a suitable inductive bias, discussing their expressive power and known theoretical limitations. Building on these foundations, I will introduce Lifted Relational Neural Networks (LRNNs) — a framework for differentiable learning over relational logic programs. LRNNs employ weighted relational rules as lifted learning templates from which neural computation graphs are dynamically constructed based on the logical models of the input data, in the spirit of ILP. I will discuss the syntax of LRNNs and their underlying semantics, showing how (fuzzy) logical connectives and inference structure are reflected in the corresponding neural computation graphs, enabling end-to-end differentiability of logic programs under a least model semantics. I will then illustrate how this logic-based formulation can recover several well-known neural architectures, including the GNNs, as special cases of weighted logical rules, while naturally extending them to richer relational settings. I will conclude by demonstrating how this supports end-to-end learning directly from relational data, while enabling advanced neuro-symbolic models with complex inductive biases, expressed in the interpretable language of relational logic.

Accepted Talks

Containment of Conjunctive Queries With Negated Atoms for Databases over Semirings

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For two queries P and Q, the containment problem asks whether (in all databases) the answers of P are contained in those of Q. This is a fundamental theoretical problem that directly connects to query optimization. Conjunctive queries (CQ) are first-order formulas with a string of existential quantifiers at the front followed by a quantifier-free matrix where the only connective used is conjunction. CQs with equations and disequations further allow the presence of formulas of the form $x = y$ and $x \neq y$ in the quantifier free matrix. Finally, CQs with negated atoms allow the presence of formulas of the form $\neg R(\overline{x})$.

The first problem this talk looks at is containment for CQs with equations and disequations over databases with relations annotated with elements of a commutative semiring (while the operations $\cdot$ and $+$ are used to interpret $\wedge$, $\vee$ and $\exists$). The latter kind of databases has received a considerable level of attention in the past two decades. Containment for regular CQs in this framework was studied in [2]. We use ideas from that paper to establish complexity bounds for the containment problem of CQs with equations and disequations for distributive lattices, lineage, why-provenance, and provenance polynomial annotations, among many others. For example, this problem over the semiring of natural numbers is undecidable [1] but we show it is in Π_2^p for the case of provenance polynomial annotations (i.e. the semiring of polynomials over the natural numbers).

The second problem we look at is containment for CQs with negated relational atoms. Negation is a significant challenge in the semiring context and we choose to explore the approach in [3], using the technical notion of 'interpretations' (basically mappings from literals to the semiring in question). We show that choosing different classes of interpretations to handle negation makes a substantial difference for the complexity of containment. For example, on distributive lattices, the problem is NP-complete or Π_2^p-complete, depending on what class of interpretations we consider. For the most basic interpretations considered in [3] for provenance semirings, we show the complexity remains the same as for regular CQs.

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On the structure of residuated po-semigroups as models of Lambek Calculus and MLL

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We are interested in finite algebraic models of multiplicative linear logic MLL, the fragment of linear logic with only the multiplicative connectives. MLL has the finite model property, and its entailment is known to be NP-complete [6]. As free MLL algebras are infinite and not so well understood, we concentrate on finite algebras, which encompass all the counterexamples. The MLL algebras are partially ordered algebras (the partial order generated by its sequent calculus, and the multiplication order-preserving), but because lattice connectives are not present in the signature, the underlying order does not need to form a (semi)-lattice, and some interesting posets (including in particular certain unions of chains) arise this way. With this motivation in mind, we aim at structural understanding of certain classes of partially ordered involutive residuated magmas/semigroups/monoids.

An involutive partially ordered magma (ipo-magma) $(A, \leq, \cdot, \sim, -)$ is a poset $(A, \leq)$ with a binary operation $\cdot$, two unary order-reversing operations $\sim, -$ that are an involutive pair: $\sim{-x} = x = -{\sim x}$, and for all $x, y, z \in A$

$$\text{(rotate)} \quad xy \leq z \iff y \cdot {\sim z} \leq {\sim x} \iff -z \cdot x \leq -y.$$

Any such algebra is a residuated partially ordered (rpo-)magma, with the two residuals arising by $xy \leq z \iff x \leq -(y \cdot {\sim z}) \iff y \leq {\sim(-z \cdot x)}$. An associative ipo-magma is an ipo-semigroup. An ipo-monoid is a unital ipo-semigroup $(A, \leq, \cdot, \sim, -, 1)$, where we can define $0 = {\sim 1} = -1$ and, as shown in [5], (rotate) can be replaced by

$$\text{(lin)} \quad x \leq y \iff x \cdot {\sim y} \leq 0 \iff -y \cdot x \leq 0.$$

Although rpo-magmas are very general, we observe that residuation imposes restrictions on the posets that can occur: (i) In an rpo-magma every connected component of $\leq$ is up-directed and down-directed, hence for finite rpo-magmas every connected component is bounded; (ii) The equivalence relation on a poset that has each connected component as an equivalence class is a congruence on a rpo-magma, and the quotient algebra is a quasigroup with the discrete order

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(i.e. $\leq$ is the equality relation), and if the multiplication is associative, it is a group. Conversely, from any group or quasigroup Q, and a pairwise disjoint family of bounded posets A_q for $q \in Q$, one can construct an rpo-magma with poset $\bigcup_{q \in Q} A_q$.

Now, any finite residuated (and thus also any finite involutive) semigroup is based on a poset like that, having connected components with top and bottom elements. A case of special interest is when the poset is a union of chains. Algebras of this form arise as subalgebras of a product $\mathbf{C} \times \mathbf{G}$ of a residuated monoid $\mathbf{C}$ with a pointed group $\mathbf{G}$. In the special case when $\mathbf{C}$ is an involutive po-monoid with a single connected component, $\mathbf{C} \times \mathbf{G}$ is an involutive po-monoid based on a union of $|G|$ many components $C_g = \{(a, g) \mid a \in C\}$. Our aim is to describe all subalgebras of these products, and in some cases, prove that all finite algebras are isomorphic to subalgebras of this form. This provides a representation theorem for certain classes of involutive monoids.

We concentrate on some specific cases: for instance, we describe algebras with the Sugihara monoid S_k as the unital component. Plonka sums described in [1] are used to show these algebras can be constructed from groups with the antichain order and ipo-monoids that are disjoint unions of one-element and two-element chains. This closely relates to work on unilinear residuated lattices [4,8], only we do not assume commutativity in general, we use involutive rather than 1-involutive linear negations, and also cover even-sized Sugihara monoids.

Some of the results were discovered with the help of Prover9/Mace4 [7].

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Termination, Countermodels, and Complexity in Bilateral Labeled Sequent Calculi

Fabio De Martin Polo*

This talk (based on [1]) investigates the proof theory of contra-classical logics, with a focus on Heinrich Wansing's constructive connexive logic C [4]. Contra-classical systems are neither subsystems nor extensions of classical logic [2]. Among them, connexive logics stand out as they include certain non-theorems of classical logic as characteristic theses, namely *Aristotle* and *Boethius theses*:

$$\sim(\sim A \rightarrow A) \text{ and } \sim(A \rightarrow \sim A); \tag{AT}$$

$$(A \rightarrow B) \rightarrow \sim(A \rightarrow \sim B) \text{ and } (A \rightarrow \sim B) \rightarrow \sim(A \rightarrow B). \tag{BT}$$

As can be readily observed, these formulas are not valid in classical logic.

Additionally, connexive systems require that the formula $(A \rightarrow B) \rightarrow (B \rightarrow A)$ be unprovable; that is, they require a *non-symmetric* implication.

Wansing observed that, in order to obtain a connexive system, the schema $\sim(A \rightarrow B) \leftrightarrow (A \wedge \sim B)$ cannot capture the falsification conditions of a connexive implication, and that these conditions instead call for the interpretation given by $(A \rightarrow \sim B) \leftrightarrow \sim(A \rightarrow B)$.

At the semantic level, C is defined through a *bilateral relational semantics* built over reflexive and transitive frames, with the connectives characterised in terms of "support of truth" and "support of falsity" at a state [4].

In this talk, drawing on this semantic bilateral framework, we introduce a *bilateral labeled sequent calculus* – denoted GC^{+}_{-} – incorporating specific "verification" and "falsification" rules (cf. Table 1). For $\bullet \in \{+, -\}$, a *bilateral labeled sequent* is a syntactic object of the form $\Lambda = \mathsf{S}_1 \Rightarrow \mathsf{S}_2$, where S_1 and S_2 are defined via the following grammars:

$$\mathsf{S}_1 ::= x :^{\bullet} A \mid x \leqslant y \mid \mathsf{S}_1, \mathsf{S}_1 \qquad \mathsf{S}_2 ::= x :^{\bullet} A \mid \mathsf{S}_2, \mathsf{S}_2$$

with A being a C formula and x, y belonging to a denumerable set of labels $\mathsf{Lab} = \{x, y, ...\}$. We refer to formulas of the form $x :^{\bullet} A$ and $x \leqslant y$ as labeled formulas and relational atoms, respectively.

We will show that all logical and relational rules are height-preserving invertible, that the structural rules are height-preserving admissible, and that the CUT rules (see Table 1) are admissible. We will also argue that GC^{+}_{-} is sound and complete with respect to C's bilateral semantics.

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$$(y \text{ fresh})\ \dfrac{x \leqslant y,\ y :^{+} A, \Gamma \Rightarrow \Delta,\ y :^{+} B}{\Gamma \Rightarrow \Delta,\ x :^{+} A \to B}\ R^{+}_{\to} \qquad \dfrac{x \leqslant y,\ x :^{+} A \to B, \Gamma \Rightarrow \Delta,\ y :^{+} A \quad y :^{+} B,\ x \leqslant y,\ x :^{+} A \to B, \Gamma \Rightarrow \Delta}{x :^{+} A \to B,\ x \leqslant y, \Gamma \Rightarrow \Delta}\ L^{+}_{\to}$$

$$(y \text{ fresh})\ \dfrac{x \leqslant y, y :^{+} A, \Gamma \Rightarrow \Delta, y :^{-} B}{\Gamma \Rightarrow \Delta, x :^{-} A \to B}\ R^{-}_{\to} \qquad \dfrac{x \leqslant y, x :^{-} A \to B, \Gamma \Rightarrow \Delta, y :^{+} A \quad y :^{-} B, x \leqslant y, x :^{-} A \to B, \Gamma \Rightarrow \Delta}{x :^{-} A \to B, x \leqslant y, \Gamma \Rightarrow \Delta}\ L^{-}_{\to}$$

$$\dfrac{x \leqslant x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \textsc{ref} \qquad \dfrac{x \leqslant z, x \leqslant y, y \leqslant z, \Gamma \Rightarrow \Delta}{x \leqslant y, y \leqslant z, \Gamma \Rightarrow \Delta}\ \textsc{trs}$$

$$\dfrac{\Gamma \Rightarrow \Delta, x :^{+} A \quad x :^{+} A, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}\ \textsc{cut}^{+} \qquad \dfrac{\Gamma \Rightarrow \Delta, x :^{-} A \quad x :^{-} A, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}\ \textsc{cut}^{-}$$

Table 1: Rules for $\to$, relational rules and CUT rules

Once both the structural analysis and completeness are established, we will turn to the core of the talk, showing that GC^{+}_{-} supports terminating proof search and enables countermodels to be extracted from failed derivations [3].

Firstly, we will show that the weak subformula and subterm properties bound the space of formulas and labels appearing during proof search; in particular, applications of relational rules – REF and TRS – will be limited to labels already present in the end-sequent or to new labels introduced by some logical rule, particularly $R^{+}_{\to}$ and $R^{-}_{\to}$.

Secondly, another source of potentially non-terminating derivations is given by the so-called contraction-absorbing rules, namely those rules whose premises contain copies of the end formula (i.e., $L^{+}_{\to}, L^{-}_{\to}$). To ensure termination, we guarantee that such rules can be applied at most once to the same pair of principal formulas along any branch.

Thirdly, after having established the bound on the number of applications of contraction-absorbing rules, we will strengthen the termination results by introducing a notion of saturated sequents and showing that, in GC^{+}_{-} proof search, such sequents can always be reached in finitely many steps. We apply the rules root-first until a suitable saturation condition is met. Intuitively, a branch is considered *saturated* when its leaf is not an initial sequent and is closed under all the rules of GC^{+}_{-} – except for applications of rules that would generate loops modulo label substitution (i.e., $R^{+}_{\to}, R^{-}_{\to}$). In this latter situation, we define a partial order by taking the reflexive and transitive closure of $\leqslant$, extended with a relation that captures the looping behavior implicit in certain rule applications. Together with the single-shot use of $L^{+}_{\to}$ and $L^{-}_{\to}$, this saturation strategy will be shown to block both looping and label duplication, ensuring a finite search space and enabling countermodels to be extracted from saturated failed derivations.

In conclusion, we determine the computational complexity of the decision procedure for GC^{+}_{-}. In particular, we show that the decision problem for the logic C is solvable in single-exponential time and space with respect to the size of the subformula closure of the input sequent.

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Syntactic vs Semantic Consistency of a Hyperintensional System with Procedural Semantics

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In this contribution, I present results obtained in collaboration with my co-author, Bjørn Jespersen, which were published in the Annals of Pure and Applied Logic. The paper resolves inconsistencies arising from the unsafe binding of variables in a hyperintensional logical system concerned with procedures.

One thing no formal semantics wants is for its variables to involuntarily shift from occurring bound to occurring free, or vice versa. Yet exactly that is liable to happen within a *procedural* semantics for no other reason than a procedure undergoes execution. This unwelcome shift is symptomatic of insufficiently regimented coordination among procedures and thus points to a flaw in the foundations of the theory in question. This flaw, in turn, jeopardises the validity of lambda conversions and the semantic evaluation of procedures. I address and solve one such set of problems in which the execution of a procedure causes bound occurrences of variables to become free, or free occurrences of new variables to arise.

I will be working within Transparent Intensional Logic (TIL), which is one of the most conceptually elaborate and technically sophisticated theories of hyperintensionality. The hyperintensional entities of TIL are so-called *procedures*. Procedures are not reducible to set-theoretical aggregates of their proper parts. The elements of an aggregate lack direct connection *inter se*, even when they are lined up in a sequence, and the sequence cannot be executed. Each kind of procedure can be *executed* as a whole to yield at most one object as its product, and each kind of procedure can *itself figure as a unit*, on which other procedures operate.

The technical problems we solve concern *variable binding* and *substitution*. A pair of twin procedures, called in TIL *Trivialization* and *Double Execution*, work well together in most cases. But there are limiting cases where they fail to. Trivialization, if applied to another procedure, makes it feasible to operate on this procedure itself rather than on the product it is typed and structured to yield. Any occurrences of variables within the procedure become Trivialization-bound. Trivialization-binding is stronger than λ-binding. Double Execution, when applied to another procedure, is typed and structured to, first, obtain the product of this procedure and, second, obtain the product of this product on condition that the latter is itself a procedure. Double Execution can turn some Trivialization-bound occurrences of variables into free occurrences, and Double Execution may also produce fresh variables. Such cases undermine the definition of substitution, thus jeopardising the validity of the λ-conversion rules. The interaction of these twin procedures must be properly coordinated in order for the whole system to be consistent and the computation of procedures to be correct.

TIL, as a system with a procedural semantics, comes with two notions of computation, namely syntactic β-conversions, and semantic evaluation of the product of a procedure with respect to a particular valuation of free variables. By imposing plausible restrictions on syntactic computation, syntactic consistency of the system is achieved, and therefore, only appropriate procedures are passed on for semantic evaluation. As a result, the problems of bound occurrences becoming free and new variables cropping up can no longer be generated. The interplay between the twin procedures of Trivialization and Double Execution makes it possible to operate on procedures occurring hyperintensionally. By applying the *substitution method*, we can, first, modify procedure C and, second, execute it. In TIL semantics, we have: ${}^2[{}^0\text{Sub}\ [{}^0\text{Tr}\ A]{}^0x{}^0C(x)]$, and the following happens.

1. *First execution* breaks down into these steps:
   (a) Execute procedure A to obtain its product, the value a; if A is v-improper, then the whole procedure is v-improper (stop); else:
   (b) $[{}^0\text{Tr}\ A]$: obtain the *Trivialization* of (or the pointer at) argument a produced by A.
   (c) $[{}^0\text{Sub}\ [{}^0\text{Tr}\ A]{}^0x{}^0C(x)]$: substitute correctly this pointer at a for x into the body C.
2. *Second*, C now has a concrete value for x, and is properly parametrised. Hence, execute this adjusted C.

The problem we address and solve extends beyond TIL. Any logic or programming language (especially if based on λ-calculus) furnished with a high degree of expressive power, in which procedures can occur in two modes, namely *execution* vs *displayed* (as operands) mode, is liable to confront similar problems. For instance, in Computational λ-Calculus (CLC), the TIL twin procedures of Trivialization and Double Execution are modelled by means of *monads* and *staging*. Similarly to TIL, we must distinguish between the code (of a procedure) and its execution. The evaluation and sequencing of computations are expressed using *monadic bind*, typically written: $x \leftarrow A; C(x)$. So, substitution and execution in TIL goes over into monadic bind in CLC. The two stages are:

- *Eval*: run A, get a result, plug the result into C.
- *Apply*: run the adjusted C.

As for variables, it is semantically essential in TIL that Trivialization displays procedures that are not open to external substitution, unless explicitly ‘opened’ by the substitution method. All the variables occurring in a displayed procedure are bound by Trivialization. CLC shares this idea: a displayed procedure is represented by its code, which is just syntax, and no substitution happens prior to evaluation. Both in TIL and CLC, variables inside displayed (quoted) procedures retain their scope and remain bound. The dual act of selection-cum-execution must not open them up in an unsafe way. The common tenet we adhere to is that it must be determinate and known at the syntactic level prior to any computation being launched, which variables occur free for substitution; if it is not followed, bad staging might lead to variable capture or leakage.

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Deductive Interpolation in Hájek's Basic Fuzzy Logic

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As chronicled in his 1998 monograph [9], Petr Hájek introduced his basic system of fuzzy logic **BL** as a means of systematizing the diverse array of work on mathematical fuzzy logic going on at that time. The importance of **BL** to fuzzy logic writ large was almost immediately apparent and, not long after its introduction, it was shown in [5] that **BL** is precisely the logic of continuous triangular norms. The equivalent algebraic semantics of **BL**—a class of residuated algebraic structures called *BL-algebras*—was also identified and subjected to extensive scrutiny. Aglianò and Montagna gave an elegant structure theory for BL-algebras in [2], proving that each totally ordered BL-algebra (the class of which is characteristic for **BL**) is an ordinal sum of 0-free MV-algebras together with an MV-algebra component. This decomposition has proven an extraordinarily powerful tool in the study of **BL** and, due in large part to this structure theory, we now know the answers to most of the fundamental logical questions one might ask about **BL**: We have a good description of its axiomatic extensions [1], have transparent relational semantics for it [7], have classifications of which of its extensions have Beth definability [10], and so forth.

One question that has proven frustratingly resistant to analysis via Aglianò-Montagna structure theory, however, is the question of interpolation in extensions of **BL**. Montagna showed in [10] that the only axiomatic extensions of **BL** with the Craig interpolation property[1] are those that are definitionally equivalent to superintuitionistic logics.[2] On the other hand, in extensions of **BL**, interpolation for deduction is strictly weaker than Craig interpolation. Montagna showed that the deductive interpolation property[3] holds for a range of

[1]A propositional logic with an implication connective $\to$ is said to have the *Craig interpolation property* if whenever $\varphi \to \psi$ is a theorem, there exists a formula δ such that both $\varphi \to \delta$ and $\delta \to \psi$ are theorems and all propositional variables appearing in δ appear in both of φ and ψ.

[2]There are just three of these: The trivial logic, Gödel-Dummett logic, and the logic of the three-element totally ordered Heyting algebra.

[3]The definition of the *deductive interpolation property* is obtained by replacing $\to$ in the

natural extensions of **BL**, including **BL** itself, but that there are uncountably many axiomatic extensions of **BL** that do not even have deductive interpolation. Montagna further posed the problem of characterizing which axiomatic extensions of **BL** have the deductive interpolation property and, in particular, asked how many such logics there are.

Montagna's problem proved extremely difficult. Cortonesi, Marchioni, and Montagna returned to the problem in [6], where they apply techniques from quantifier elimination to study amalgamation in BL-algebras, a semantic rendition of deductive interpolation. Later on, Aguzzoli and Bianchi applied algebraic tools centering on the Aglianò-Montagna structure theory to study deductive interpolation in extensions of **BL** in [3, 4]. However, these studies failed to yield a complete resolution to Montagna's problem because of the necessity of including finiteness hypotheses for methods applied therein.

In this talk, we report on our recent solution of Montagna's problem, recorded in [8]. In particular, we give an exhaustive description of all axiomatic extensions of **BL** with the deductive interpolation property. Like many previous efforts, our solution focuses on the analysis of amalgamation in varieties of BL-algebras. However, unlike previous attempts, our effort is bolstered by new general-purpose results on the amalgamation property that can be applied profitably to the study of BL-algebras. In the end, we provide a tangible naming scheme inspired by the theory of regular expressions that concretely identifies all varieties of BL-algebras with the amalgamation property, showing that these can be partitioned into countably infinitely many finite intervals. This shows, among other things, that there are only countably many axiomatic extensions of **BL** with the deductive interpolation property. Our results apply also to the 0-free subreducts of BL-algebras, allowing us to obtain a similar classification for extensions of the negation-free fragment **BL**. We also discuss ramifications of this general method to the study of interpolation in substructural logics generally.

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definition of the Craig interpolation property by the consequence relation $\vdash$ of the logic in question. In full, a logic with consequence relation $\rightarrow$ has the deductive interpolation property if whenever $\varphi \vdash \psi$ holds, there exists a formula δ such that both $\varphi \vdash \delta$ and $\delta \vdash \psi$ hold and all variables appearing in δ appear in both φ and ψ.

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Paraconsistent Constructive Modal Logic

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Constructive modal logics are characterized by their simple proof-theoretic presentation in terms of (cut-free) Gentzen sequent calculi (cf., e.g., [7, 6, 2]). Their semantics can be specified in terms of bi-relational Kripke models containing both a pre-order, as in Kripke frames for propositional intuitionistic logic (Int), and an accessibility relation. The main difference with other proposals such as IK by Fisher-Servi [4] and Simpson [6] is that no properties are assumed to relate the pre-order and the accessibility relation. Both Wijesekera's logic and constructive modal logic CK satisfy some of the conditions stated by Simpson [6]: they are conservative extensions of Int, they have the disjunction property, and the two modalities are independent. Notice that the hereditary condition, necessary to ensure the conservativity over Int is built in by the forcing conditions, without the need for specific frame conditions (as, e.g., is done in IK, recent **FIK** [1] and Došen's $\mathbf{HK}_{\Box}$ [3]).

In this work, we aim to define a paraconsistent counterpart of CK. The logic we are seeking must have the following features:

1. it has the property of constructive falsity: if $\neg(\phi \wedge \chi)$ is provable then either $\neg\phi$ or $\neg\chi$ is provable;
2. $\phi \wedge \neg\phi$ does not entail every proposition (paraconsistency);
3. contradictions are not equivalent: $p \wedge \neg p$ is not equivalent to $q \wedge \neg q$.

Intuitionistic logic with its standard negation $\neg$ satisfies none of the three conditions. Our starting point for the interpretation of propositional connectives is the well-known logic N4 by Nelson [5]. In this logic, intuitionistic negation and falsity are replaced by so-called strong negation $\sim$ that satisfies the properties above. On the other hand, paraconsistent modal logics provide a more intuitive doxastic and deontic interpretations of modalities. Classically (and intuitionistically), to account for contradictory beliefs or obligations, one may consider a non-normal or non-regular logic. In the first case, $\Box(\phi \wedge \chi)$ is not equivalent to $\Box\phi \wedge \Box\chi$. In the second case, $\Box\phi \to \Box\chi$ does not follow from $\phi \to \chi$. Still, even in these cases, all contradictions are equivalent. Hence, if an agent believes in one contradiction, they believe in *all contradictions*. If an agent has one conflicting obligation, then all obligations are contradictory. In addition to that, even in the presence of contradictory beliefs and obligations, one might want to utilise characteristic features of normal and regular modalities. Both options are possible when using paraconsistent logics.

The aim of this work is to define a family of paraconsistent constructive modal logics of increasing strength, all of which can be considered $\mathsf{N4}$-like counterparts of CK in a loose sense. We get several systems from the weakest to strongest according to the relation between the two modal operators $\Box$, $\Diamond$ and their strong negations: $\sim\Box$, $\sim\Diamond$. In the weakest system, there is no relation among the four. In the strongest one, $\Box$ and $\Diamond$ are reducible to one another via strong negation. The corresponding semantic picture is to consider $\mathsf{N4}$-models having one or more accessibility relations for defining the modal operators. Namely, the models of the weakest system contain *four* independent accessibility relations (one for each modality and for their negations); the models of the strongest logic interpret all modalities using the same relation. We provide strongly complete Hilbert axiomatizations for all these logics and construct cut-free sequent calculi which we use to establish the decidability of our systems.

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Hybrid Logic of Strict Betweenness

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The purpose of this talk is to present a hybrid logic for the ternary geometric betweenness relation in the sense of (Borsuk and Szmielew, 1960).

Let $\mathfrak{F} \coloneqq \langle U, B \rangle$ be a 3-frame, i.e., a frame with a ternary relation on the set of points. We will read $B(x, y, z)$ as *y is between x and z*. $\#(x, y, z)$ means that the elements of the set x, y and z are pairwise different. The betweenness axioms we take into account are:

$$B(x,y,z) \to \#(x,y,z)\,, \tag{B1}$$

$$B(x,y,z) \to B(z,y,x)\,, \tag{B2}$$

$$B(x,y,z) \to \neg B(x,z,y)\,, \tag{B3}$$

$$B(x,y,z) \wedge B(y,z,u) \to B(x,y,u)\,, \tag{B4}$$

$$B(x,y,z) \wedge B(y,u,z) \to B(x,y,u)\,, \tag{B5}$$

$$\#(x,y,z) \to B(x,y,z) \vee B(x,z,y) \vee B(y,x,z)\,, \tag{B6}$$

$$x \neq z \to \exists y\, B(x,y,z)\,, \tag{B7}$$

$$\forall y \exists x \exists z\, B(x,y,z)\,. \tag{B8}$$

Any 3-frame $\mathfrak{F}$ that satisfies (B1)–(B5) will be called *a betweenness frame* or simply a *b-frame*. The class of all b-frames will be denoted by '**B**'. We also distinguish the following classes:

(1) **LB** $\coloneqq$ **B** + (B6) of *linear* b-frames,

(2) **DLB** $\coloneqq$ **B** + (B6) + (B7) of *dense* linear b-frames,

(3) **LBWE** $\coloneqq$ **B** + (B6) + (B8) of *linear* b-frames *without endpoints*,

(4) **DLBWE** $\coloneqq$ **B** + (B6) + (B7) + (B8) of dense linear b-frames without endpoints.

As the betweenness relation is ternary for its modal analysis we are going to need binary modal operators. The basic idea for such an operator $\langle B\rangle$ comes from van Benthem and Bezhanishvili (2007). Given a model $\mathfrak{M} \coloneqq \langle \mathfrak{F}, V\rangle$ based on a b-frame $\mathfrak{F}$ we characterize the semantics for $\langle B\rangle$ in the following way

$$\mathfrak{M}, w \Vdash \langle B\rangle(\varphi,\psi) \;:\!\longleftrightarrow\; (\exists x, y \in W)\,(\mathfrak{M}, x \Vdash \varphi \text{ and } \mathfrak{M}, y \Vdash \psi \text{ and } B(x,w,y))\,. \tag{df $\langle B\rangle$}$$

$\langle B\rangle$ gives rise to a natural unary *convexity* operator

$$\mathrm{C}\,\varphi \;:\!\longleftrightarrow\; \langle B\rangle(\varphi,\varphi)\,. \tag{df C}$$

As has already been said, we are going to study the properties of $\langle B\rangle$ in the hybrid language with two sorts of variables: *propositional letters* p, q, r and so on, and *nominals* i, j, k, l, indexed if necessary. The set of all propositional letters will be denoted by 'Prop', and the set of nominals by 'Nom'. We assume that $\mathrm{Prop} \cap \mathrm{Nom} = \varnothing$. The *valuation* function is any function $V\colon \mathrm{Prop} \cup \mathrm{Nom} \to \mathcal{P}(U)$ such that for every nominal i, $V(i)$ is a singleton subset of the universe.

Recall that the semantics of the *at* operator—for which we standardly use @—is given by the following

$$\mathfrak{M}, w \Vdash @_i\,\varphi \;:\!\longleftrightarrow\; \mathfrak{M}, V(i) \Vdash \varphi\,. \tag{df $@_i$}$$

We can see that

$$\mathfrak{M}, w \Vdash \langle B\rangle(i,j) \longleftrightarrow (\exists x \in U)(\exists y \in U)\,(V(i) = \{x\} \text{ and } V(j) = \{y\} \text{ and } B(x,w,y))\,.$$

Using the standard techniques from (Blackburn et al., 2001) and generalized tools from (ten Cate, 2005), we are going to prove that

Theorem 0.1. *For every $i \in \{1,3,4,\ldots,7\}$ the class of frames that satisfies* (Bi) *is not modally definable. Moreover, the class of* (B7)*-frames is not @-definable.*

Theorem 0.2. **DLBWE** *is @-definable. Indeed,* **DLBWE** *is @-definable by pure formulas.*

Making use of some results from (Bezhanishvili et al., 2023) we also show that

Theorem 0.3. *For any 3-frame $\mathfrak{F} \in$* **LBWE**: *$\mathfrak{F} \vDash$ (B7) iff $\mathfrak{F} \Vdash \mathrm{C}\,p \to \mathrm{C}\,\mathrm{C}\,p$, i.e., density is modally definable with respect to the class* **LBWE**.

We also discuss a system of hybrid logic L_B of strict betweenness built in the language with @-operator (following the style of ten Cate, 2005, Definition 5.1.2) and we prove its completeness with respect to the class of countable dense linear orders without endpoints.

Finally, we analyze the case of the real line treated as the model of betweenness with the second-order completeness axiom, and we put forward a system of hybrid logic with the universal modality complete w.r.t the real line.

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Formalization of set theory fragments in Isabelle/HOL

Štěpán Holub and Zuzana Haniková

The talk will report on our formalization of different collections of axioms for the set theory in Isabelle/HOL [3]. Our original motivation is Alternative Set Theory (AST) of Petr Vopěnka [2]. Since AST does not admit infinite sets, its axioms are closely related to different axiomatizations of theory of hereditarily finite sets. One purpose of our formalization is to provide an accessible, extendable and verified collection of known results scattered throughout the literature. They include in particular dependence and independence of various axioms, and minimal environments allowing to prove several claims. A prominent example is the fact that scheme of regularity does not follow from other axioms (including the "ordinary" regularity axiom) if the axiom of infinity is negated.

Another purpose is to investigate possibilities of formalization of nonstandard extensions and variants of ZFfin, including in particular the axiom of existence of semisets. While Vopěnka was often dismissive of formalized aspect of mathematics in general, and of AST in particular, it should be stressed that he eventually uses standard logical apparatus. Formal verification may help to highlight the indisputable core of the theory and to isolate its more vague corners.

A crucial part of the proposed contribution will discuss the question of how the goal outlined above is compatible with the use of the prove assistant Isabelle, and its HOL variant. A care is needed to ensure that the strong higher order logic does not invalidate the study of weak theories in question. Our approach is based on two design decisions: 1.We model fragments of theories by *locales*, which makes sure that obtained results are valid for any instantiation. Conversely, missing inferences can be proven in the usual model-theoretic way by constructing suitable counter-example interpretations. (For example permutation models, in case of the regularity mentioned above, following [1]) 2.We treat first order formulas quantified in axiom schemas semantically as predicates, and define set-theoretical predicates using the native inductive definition mechanism of Isabelle/HOL.

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Rule-Dependence and -Independence in Meaning Constituting Rules

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According to a view often attributed to Wittgenstein meaning is constituted by rules. Currently one of the most influential way to develop this idea is known as inferentialism, according to which the rules that constitute meaning are inferential rules.[1] In this talk I will argue that there is a significant issue with these rules-based accounts.

In the literature on constitutive rules, there are two types of rules which, I argue, take a fundamentally different approach to meaning-constitution. Rules of the first type determine which expressions can be used to express certain semantic content. These rules are what could be called definitional-rules and they can usually be stated using a locution of a form: in context C, X-counts as Y. Such rules determine that a rule-independent act X counts as a rule-dependent act Y. For example "in German uttering "Gift" counts as expressing the concept *poisonous*. However, since the Y-term contains an essential reference to the concept *poisonous*, on the pain of circularity or regress, it cannot determine what that concept is. That is, it can only determine the rule-independent means of expressing the concept *poisonous*, but not what it means to express it.

Rules of the second type determine the conditions under which the use of an expression is permitted, and these conditions also determine the meaning of that expression. These rules could be called deontic rules. A comparison between truth-conditional semantics and rule-based inferential semantics will help to clarify this option. According to the former, the meaning of an expression "Gift" is determined by the manner the expression effects the truth-conditions of the sentence containing it. According to the latter, the meaning of an expression "Gift" is determined by how it may and may not be used in making assertions and what further assertions are licensed by the assertions expressed by that expression. In short, the role of truth conditions play in the truth-conditional semantics are replaced by permissibility conditions.

However, following Randsel [5] and Hindriks [3]. I divide deontic-rules into two types, XZ and YZ, based on whether the conditions of permissibility (Z)

[1] see e.g. [2, 4]. for non-inferential approaches to rule constitution view of meaning see e.g. [1, 6]

are attached to rule-independent acts (X) or rule-dependent acts (Y). Rule-independent acts relevant to the question of meaning are phonetic acts, singular acts of making a certain kind of noise. Assuming that meaning is constituted by rules, the rule-depended acts could be expressing certain concept, or asserting a certain proposition (locutionary act). As we saw earlier, definitional rules could arguably determine which phonetic act could in principle determine which phonetic act counts as a locutionary act, but fail to determine the content of that locutionary act. In contrast, even if we assuming that the YZ-rules can determine the the content of locutionary acts, they fail to determine a rule-independent means of performing those acts.

I examine whether the issue could be avoided by adopting XZ-rules, which attach deontic consequences directly to rule-independent terms. However, I argue that because the meaning of an utterance is underdetermined by its rule-independent physical features, XZ-rules encounter problems in determining which Y-terms are to replace which X-terms. Finally, I show that combining counts-as rules and YZ-rules cannot escape these issues. If a counts-as-rule can determine that a rule-independent X-act, such as making a particular sound, is a rule-dependent Y-act, such as the expression of a concept or the use of an expression, it must also be able to determine the meaning of that particular sound. However, the meaning was to be determined by whatever YZ-rules govern the use of that sound, but to be governed by YZ-rules, this particular utterance must count as a Y act. Therefore, circularity ensues.

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Compactness for Small Cardinals in Mathematics: Principles, Consequences, and Limitations

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We discuss some well-known compactness principles for uncountable structures of small regular sizes (ω_n for $2 \leq n < \omega$, $\aleph_{\omega+1}$, $\aleph_{\omega^2+1}$, etc.), consistent from weakly compact (the size-restricted versions) or strongly compact or supercompact cardinals (the unrestricted versions). We divide the principles into *logical principles*, which are related to cofinal branches in trees and more general structures (various *tree properties*), and *mathematical principles*, which directly postulate compactness for structures like groups, graphs, or topological spaces (for instance, countable chromatic and color compactness of graphs, compactness of abelian groups, Δ-reflection, Fodor-type reflection principle, and Rado's Conjecture).

We also focus on *indestructibility*, or *preservation*, of these principles in forcing extensions. While preservation adds a degree of robustness to such principles, it also limits their provable consequences. For example, several well-known mathematical problems decided by $V = L$ and by forcing axioms, in the opposite ways, i.e. Suslin Hypothesis, Whitehead's Conjecture, Kaplansky's Conjecture, and Baumagartner's Axiom, are independent from some of the strongest forms of compactness at ω_2. This is a refined version of Solovay's theorem that large cardinals are preserved by small forcings and hence cannot decide many natural problems in mathematics. Additionally, we observe that Rado's Conjecture plus $2^\omega = \omega_2$ is consistent with the negative solutions of some of these conjectures (as they hold in $V = L$), verifying that they hold in suitable Mitchell models.

Finally, we comment on whether the compactness principles under discussion are good candidates for axioms. We consider their consequences and the existence or non-existence of convincing unifications (such as Martin's Maximum or Rado's Conjecture). This part is a modest follow-up to the articles by Foreman "Generic large cardinals: new axioms for mathematics?" (1998) and Feferman et al. "Does mathematics need new axioms?" (2000).

Interpolation Properties Among Arbitrary Extensions of RM

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The most well-known results on interpolation properties among relevance logics have came down to showing the failure of this property [8, 9]. Despite this, there are number of overlooked instances where relevance logics enjoy various interpolation properties. In this work, we add to the storehouse of such affirmative examples by giving a complete classification of arbitary (not just axiomatic) extensions of the logic **RM** with the following strong form of deductive interpolation, often called the *Maehara Interpolation Property* (or **MIP**):

If $\mathsf{var}(\Sigma \cup \{\alpha\}) \cap \mathsf{var}(\Gamma) \neq \emptyset$ and $\Sigma, \Gamma \vdash \alpha$, there exists a set of formulas Δ such that $\mathsf{var}(\Delta) \subseteq \mathsf{var}(\Sigma \cup \{\alpha\}) \cap \mathsf{var}(\Gamma)$ and both $\Gamma \vdash \Delta$ and $\Sigma, \Delta \vdash \alpha$. (**MIP**)

We show that among all extensions of **RM**, there are exactly five logics with the **MIP**.

This work traces a tradition in relevance logic going all the way back to its beginnings. Indeed, Anderson and Belnap [1] showed already in the 1970s that the logic of first-degree entailment has the so-called *perfect Craig interpolation property* (**PCIP**) in the following form:

If $\vdash \alpha \to \beta$, then there is a formula δ such that $\mathsf{var}(\delta) \subseteq \mathsf{var}(\alpha) \cap \mathsf{var}(\beta)$ and both $\vdash \alpha \to \delta$ and $\vdash \delta \to \beta$. (**PCIP**)

It was shown in [1, pp. 416-417] that **RM** lacks not just the **PCIP**, but also the *Craig interpolation property* (**CIP**) in the following imperfect form:

If $\nvdash \neg\alpha$, $\nvdash \beta$, and $\vdash \alpha \to \beta$, then there is a formula δ such that $\mathsf{var}(\delta) \subseteq \mathsf{var}(\alpha) \cap \mathsf{var}(\beta)$ and both $\vdash \alpha \to \delta$ and $\vdash \delta \to \beta$. (**CIP**)

On the other hand, adding the truth constatns **t** and **f** to **RM** makes quite a difference in terms of interpolation properties: $\mathbf{RM^t}$ has the CIP as has been shown by Meyer in [7]. Futhermore, it is well known [5, 4] that in case of

$\mathbf{RM^t}$, **CIP** entails the *deductive interpolation property* (**DIP**), obtained from the **MIP** by taking the special case where $\Sigma = \emptyset$:

> If $\alpha \vdash \beta$, then there is a formula δ such that $\mathsf{var}(\delta) \subseteq \mathsf{var}(\alpha) \cap \mathsf{var}(\beta)$ and both $\alpha \vdash \delta$ and $\delta \vdash \beta$. (**DIP**)

Indeed, the **DIP** coincides with the **PCIP** for axiomatic extensions of $\mathbf{RM^t}$, and there are precisely nine axiomatic extensions of $\mathbf{RM^t}$ that have these two equivalent properties, as has been proven in [6]. This works further fills in this picture, showing what happens in the case without constants while focusing on interpolation for deducibility instead of implication.

In order to obtain our results, we exploit the bridge theorems of abstract algebraic logic, allowing us to examine the **MIP** via purely algebraic methods. In algebraic terms, we show that there are exactly five subquasivarieties of Sugihara algebras possessing the amalgamation property (**AP**). It turns out that those quasivarieties also have the relative congruence extension property (**RCEP**) which enables us to establish that exactly five quasivarieties of Sugihara algebras have the *transferable injections property* (**TIP**) due to the following fact, which holds for arbitrary quasivarieties:

$$\mathbf{TIP} \Leftrightarrow \mathbf{AP} + \mathbf{RCEP}$$

Knowing that **RM** is algebraizable with the (quasi-)variety of Sugihara algebras, we use the characterization obtained in [2, 3]:

MIP for a logic is equivalent to **TIP** for a corresponding quasivariety

Applying this, we conclude that exactly five extensions of **RM** have the **MIP**. As a by-product of our methods, we also obtain the surprising fact that the **MIP** and the Robinson consistency theorem are equivalent for all extensions of **RM**. Finally, we also provide Hilbert-style bases for the five logics with the **MIP**.

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Intermediate Quantifiers and Their Syllogisms in Fuzzy Natural Logic

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We will briefly mention the concept of Fuzzy Natural Logic (FNL) that is a system of theories of mathematical fuzzy logic enabling us to model special cases of human reasoning that is based on the use of natural language. FNL stems from the results of classical linguistics, logical analysis of concepts and semantics of natural language and is formalized using higher-order mathematical fuzzy logic (see [1]) with MV-algebra of truth values.

One of essential parts of FNL is a formal theory of *intermediate quantifiers* (see [2]) that are expressions of natural language, for example *Most, Almost all, Several, A few*, and other similar expressions. They have been informally introduced in [3] and are special cases of generalized quantifiers introduced in [4] and further elaborated by many logicians (cf. [5] and citations therein). Their formal theory was introduced in [6].

We will also mention reasoning using *intermediate syllogisms*, i.e., generalized syllogisms in which these quantifiers occur. They are divided into 4 figures and there exist over 4000 possible syllogisms with 5 intermediate quantifiers (*All, Almost all, Most, Many, Some*). However, only 105 of them are valid. There are 3 methods for proving validity of intermediate syllogisms: syntactic or semantic proof and verification using graded Peterson's rules. We will outline all three methods. We will show that validity of intermediate syllogisms is a consequence of two algebraic inequalities and one equality. We will also present graded square of opposition with intermediate quantifiers.

Finally, we will show that our theory is capable at solving some problems of non-monotonic logic. Note that non-monotonicity is a feature of commonsense reasoning which is characteristic by the necessity to revise given axioms when a new information is obtained. A typical example is the classical *bird-penguin problem*. A commonsense knowledge tells us that "All birds fly". We argue that in commonse reasoning we do not have in mind mathematical "All" but a commonsense knowledge "Most birds fly". Then we can prove that adding a new formula expressing that "Penguins are birds which do not fly" does not lead to a contradiction.

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Coalgebraic Dynamic Logic: Safety and Reducibility

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Propositional dynamic logic and its variations

Propositional dynamic logic (PDL) [FL79, HKT00] is a well-known modal logic for reasoning about non-deterministic programs. In PDL, programs are an explicit part of the syntax, and PDL thus combines modal reasoning with a programming language. Recently, several *many-valued variants* of PDL have been investigated, for example to argue about searching games with errors [Teh14] or costs of computations [Sed20]. Parikh's *game logic* [Par83, PP03] can be seen as a generalisation of PDL from programs to 2-player games, which allows for reasoning about program correctness in a distributed setting where the environment is viewed as an opponent. A variant of game logic has been applied to reason about hybrid systems [Pla15] and neural networks [TMP24]. While PDL is based on Kripke semantics, game logic is based on monotone neighbourhood semantics. A similar neighbourhood-based dynamic logic is found in *Instantial PDL* [vBBE19], proposed as a modal logic for computation in open systems.

All these logics have in common that they combine modal reasoning with an algebra of operations on their semantic structures. Thus, they naturally fit into the unifying framework of *coalgebraic logic*.

Coalgebraic logic

Coalgebras allow us to model *state-based transition systems* via a signature functor $\mathsf{F}\colon \mathsf{Set} \to \mathsf{Set}$. For example, this includes Kripke frames ($\mathsf{F} = \mathcal{P}$ the covariant powerset functor), neighbourhood frames ($\mathsf{F} = \mathcal{M}$ the monotone neighbourhood functor) and instantial neighbourhood frames ($\mathsf{F} = \mathcal{PP}$ the double covariant powerset functor). Formal reasoning about classes of coalgebras is the subject of *coalgebraic logic*, where modalities correspond to certain natural transformations called *predicate liftings* [Pat03, KP11]. An advantage of this setting is that it allows for *general category-theoretical proofs* which are 'parametric' in the choice of semantical structures, modalities, algebra of truth-values, *etc.*

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Indeed, PDL and game logic were abstracted into *coalgebraic dynamic logic* in [HKL14, HK15]. This work, however, is restricted to two-valued logic and relies heavily on the assumption that the signature functor F is a *monad*. Various (recent) examples of dynamic logics such as many-valued PDL or Instantial PDL (see the previous section) do not meet these requirements.

Coalgebraic dynamic logic generalised

In this talk, we present a broader framework for coalgebraic dynamic logics [HP25a, HP25b] lifting all the above-mentioned restrictions and, among others, present extensions of the main results of [HKL14] therein. Here, the signature functor F is not required to be a monad, our coalgebraic logic may have countably many modalities and furthermore may be fuzzy/many-valued (taking values in an arbitrary $\mathsf{FL}_{\mathrm{ew}}$-algebra). In particular, this allows us to encompass all of the above-mentioned dynamic logics into the coalgebraic framework.

Our coalgebraic dynamic logics are parametric in (i) the choice of the *coalgebraic signature functor* F, (ii) the choice of an algebra of *truth-values* $\mathbf{A}$, (iii) the choice of a countable collection of $\mathbf{A}$-valued *predicate liftings*, (iv) the choice of a finite collection of *coalgebra operations* (generalising, *e.g.*, non-deterministic choice, sequential composition and iteration of PDL or game logic) and (v) the choice of a finite collection of *tests*.

Safety and reducibility

A fundamental compositional aspect of dynamic logics such as PDL and game logic is that operations and tests are safe for bisimilarity [vB98, Pau00], meaning that it suffices to check bisimilarity for the atomic actions to conclude bisimilarity for all actions. For our first sample application, we discuss this topic of *bisimulation safety* in the general coalgebraic framework. We identify category-theoretical conditions for coalgebra operations and tests to be safe under bisimulation/behavioural equivalence and show that common constructs such as composition and iteration (defined via a monad or double-monad structure on F) are always safe.

We proceed to discuss the topic of *reducibility* for coalgebra operations and tests, entailing the soundness of certain reduction axioms. For one, we show that reducibility always implies safety. For finitely-valued logics, we also show that if all operations and tests are reducible, then one-step completeness of the underlying coalgebraic modal logic implies strong completeness of its dynamic version. This generalises the main results of [HKL14], and with concrete instantiations we obtain, *e.g.*, strong completeness for a many-valued variant of iteration-free game logic and for two-valued iteration-free PDL with $\mathbf{A}$-weighted accessibility relations.

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Truthmaker Semantics and Curry-Howard Correspondence

Vít Punčochář

In my talk I will connect two areas that have been developed independently: type theory and truthmaker semantics. In particular, I will present a truthmaker semantics for the typed lambda calculus and some of its variants.

Kit Fine (2017c) characterizes the basic idea of truthmaking as the idea of something on the side of the world—a fact, or a state of affairs—verifying, or making true something on the side of language—a statement or a proposition. This basic idea can be found behind the possible world semantics that has played a major role in formal analyses of sentential meaning. However, truthmaker semantics, as a specific version of situation semantics, replaces possible worlds with states that are partial and have rich mereological structure: one state can be a part of another state, states can overlap, and they can be fused into richer states. Moreover, truthmaker semantics is based on the notion of "exact verification", which requires that the verifying state must be wholly relevant to the content of the verified proposition. It can then happen that while a state a exactly verifies a proposition P, some extension b of a does not verify P because b includes as its part some content that is not relevant to P.

Building on van Fraassen's (1969) ideas, modern *truthmaker semantics* has emerged largely through the work of Kit Fine (2017a,b,c). It has been applied to various linguistic and logical phenomena such as analytic implication, subject-matter, counterfactuals, imperatives, scalar implicature, and many others (see Fine, 2017c for an overview). Truthmaker semantics for intuitionistic logic was developed in (Fine, 2014). Fine characterized it as a "cross between construction-oriented semantics of Brouwer-Heyting-Kolmogorov and the condition-oriented semantics of Kripke." Similarly to Kripke semantics, truthmaker semantics is based on a relation between states and formulas. However, due to the exact nature of the truthmaker relation, the semantic clauses strongly resemble those of the BHK interpretation of logical connectives.

The relation of truthmaker semantics for intuitionistic logic to Kripke semantics is explored in detail in (Fine, 2014). In fact, this connection is essential for the completeness proof, which is the main result of (Fine, 2014). However, the connection to the construction-oriented type-theoretic frameworks that more directly implement the BHK interpretation of logical connectives has remained somewhat unclear. The goal of my talk is to show that there is a robust link between the two areas. In particular, I will develop truthmaker semantics for the *simply typed lambda calculus with products and sums*, which corresponds to intuitionistic propositional logic via the Curry-Howard correspondence. In this framework, lambda terms of the lambda calculus are associated in a compositional way with the truthmakers of Fine's semantics. In the context of intuitionistic logic, this makes good sense, because, through the Curry-Howard correspondence, lambda terms encode proofs and derivations, and it is an essential feature of intuitionistic philosophy that it identifies truth with provability.

Ansten Klev (2017) discussed several crucial differences between truthmaker semantics and constructive type theory. Our semantic construction will show that despite these differences there is a deep internal connection between the two areas. The most significant point discussed by Klev is that truthmakers in Fine's framework form a lattice structure which does not allow us to reconstruct from which sources a particular truthmaker was "derived". For example, assume that there are two different truthmakers a_1, a_2 of φ and two different truthmakers b_1, b_2 of ψ. Then the fusion of a_1 and b_1 and the fusion of a_2 and b_2 are both truthmakers of $\varphi \wedge \psi$, but if the fusion of a_1 and b_1 is the same state as the fusion of a_2 and b_2, then it is impossible to recover unambiguously a_1 and b_1 from the fusion of a_1 and b_1, and thus it is not possible to trace back in which way the truthmaker of the conjunction was obtained. This contrasts with constructive type theory, where the structure of terms always allows us to reconstruct how they were derived.

This feature is a serious obstacle that seems to undermine the possibility of a compositional interpretation of lambda terms by Fine's truthmakers. Nevertheless, I will show that the obstacle can be overcome in an elegant way by reformulating truthmaker semantics "productively", in the sense that the semantic clauses not only operate on the elements of the underlying algebraic structure but also produce new copies of these elements. For this reason, we will call this modified semantic framework *productive truthmaker semantics*. This mechanism preserves the derivational history of a truthmaker and provides the necessary structure for a fully compositional interpretation of lambda terms.

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Inexpressible Propositions and Limits of Knowledge, Belief and Truth

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Recently several authors proved that

> (*) *There exist propositions that are inexpressible.*

Wiśniewski (2011), for example, demonstrated that if propositions are identified with *possible-worlds propositions* – as is usual in many branches of logic and Montague-style formal linguistics – then a few notions of recursion theory provide a sufficient means for proof of (*). A similar general conclusion was made also e.g. by Fritz (2018) without the direct recourse to the recursion theory.

The fact (*) can naturally be adjusted in order to fit even '*hyperintensional*' *individuation of propositions* – according to which propositions are fine-grained entities that determine possible-worlds propositions (see Raclavský 2020). Such semantic theory will be adopted in the talk, since it is rather natural: an expression E of a language L expresses (as *meaning*) a structured algorithmic procedure (see e.g. Tichý 1988, Moschovakis 2006) that determines (calculates) E's *denotatum* (be it an extension or possible-worlds intension). In case of sentences, meanings expressed by them will simply be called *propositions*, while their denotata will be truth values (an appropriate adjustment can maneuvre possible-world propositions there).[1]

The present talk (based on a detailed analysis in Raclavský 2020) demonstrates that

> (**) *The existence of inexpressible propositions affects logical analysis (i.e. explication) of key propositional notions such as belief, knowledge and truth.*

In other words, the talk shows how some of inexpressible propositions look like. All our examples involve propositions reporting an agent's attitude towards a proposition, i.e. *propositional attitudes* (also widely known as *belief atittudes*, since belief is a prominent type of such attitudes).

[1] If anyone is uncomfortable with our understanding of $1 + 2 = 3$ and Fermat Last Theorem as two distinct but congruent propositions, she is free to understand the two being one and the same set of all possible worlds, i.e. the same proposition (the price to be paid: the paradox of hyperintensionality) – and still she may follow main observations made in the talk.

In some sense, the present talk contributes to the recent interest in *intensional paradoxes* (Priest 1991) and *multimodal paradoxes* (e.g. Tucker 2018, Tucker and Thomason 2011, Bacon and Uzquiano 2018), but the root of the debate lies in the early 1960s, cf. esp. Montague (1963). To avoid misunderstanding, the Paradox of Knower (and the related group of paradoxes) is unrelated to the present topic and requires different treatment (cf. Raclavský 2020 for more). Some anticipation of our results were obtained by Prior (1961), Anderson (1983) and Tichý (1988) who argued/proved that the Liar cannot assert that he is a liar.

The inevitable allusion to paradoxes (caused by historical coincidence) is an unfortunate one. No 'Liar bussiness' and similar paradox-solving enterprizes of philosophical logic is in plan here: we do not want to show a paradox to 'solve' the paradox in this or that logical system. The talk rather conforms Tarski's ([1933] 1956) treatment of paradoxes: considering them as our tools that reveal how a proper logical analysis of key notions of our conceptual scheme should look like. We will simply follow the *Principle of Non-Contradiction* (PNC) and will exclude the inconsistent understanding of the notions.

On the other hand, that principle doesn't enforces one being 'totally classical': we will embrace the standard assumption of recursion theory (now rathr called *computable functions* theory) and thus (meta)mathematics (see e.g. the classical authority of Kleene 1952) that there are *partial functions*, in the sense of function which do not return an output (= do not have a value) for every argument in its domain.[2] (Another natural assumption of the talk is that of quantification over functions. The objection that the resulting *higher-order logic* is incomplete or that it doesn't have models etc. can be dismissed by the fact that the (type-theoretic) higher-order logics which handle both total and partial functions, eg. the logic in Raclavský 2020, do have Henkin-style completeness w.r.t. to general models, see Kuchyňka and Raclavský 2024.)

Using the above common assumptions it is not difficult to prove that (*) and (**) indeed hold. But it is important to be careful in understanding those *limits*. For example, the talk doesn't say that sentences about e.g. some sentence's lack of truth or somebody's lying are meaningless. Moreover, no principle can refrain somebody from pronouncing e.g. "I assert an untrue propositions". However, notice that the sentence has the meaning one imagines it has – but by pronouncing the sequence of sounds the speaker cannot bring about the state of our world in which she truly asserts an untrue proposition. An analogue can be found in the propositional notion of belief – which is also not governed by the T-axiom. But the notion of knowing – which, on the other hand, is governed by the T-axiom – is limited, too, since one obviously cannot know the proposition that she doesn't know that. Bold philosophical conclusions need not to be newly stated, they're already known. For example, 'All truths are knowable' is a slogan of verificationism that has already been discredited by the Church-Fitch knowability paradox – we only need to adopt the fact (cf. emphasised

[2] The existence of partial functions is experienced in main areas of *computing*: programs do not deliver the output, database requests find no record as value etc. *Linguistics* knows *non-referring expressions*. A fortiori, *mathematics* embraces partiality of the familiar arithmetical division function ($n \div 0$ is undefined for any n), other familiar functions such as minus fail to have a value in some domains, e.g. $3 - 5$ in $\mathbb{N}$, and many functions in algebra and analysis do not deliver a value too. Partiality is omnipresent. The Principle of Excluded Middle must of course be carefully adjusted.

ny Williamson 2000) that it has been proved from our basic axioms concerning knowledge and necessity.[3]

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[3] Even this abstract is limited: a nice line of reasoning shows (*) and (**) as a consequence of Cantor Theorem.

Modal Weak Kleene Logics Through Variable Inclusion

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Weak Kleene logics, traditionally distinguished by an *infectious* third value (2) alongside the classical values 1 and 0, provide a robust framework for analyzing semantic nonsensicality, computer errors, and topic sensitivity. The properties of the propositional fragments of these logics are well-established (e.g., [3, 2, 10, 5]): the paracomplete weak Kleene logic ($\vDash^{\mathbf{ss}}$, preserving 1) and the the paraconsistent weak Kleene logic ($\vDash^{\mathbf{tt}}$, preserving 1 and 2) are characterized by classical logic augmented with variable inclusion constraints. Specifically, $\vDash^{\mathbf{ss}}$ corresponds to the fragment of classical logic preserving occurrences of propositional variables from conclusions to premises (provided the premises are classically consistent), while, dually, $\vDash^{\mathbf{tt}}$ corresponds to the fragment of classical logic preserving occurrences of variables from premises to conclusions (provided the conclusion is not a classical theorem).

However, the modal extension of weak Kleene logics remains relatively unexplored. Extant literature offers only a few exceptions [6, 4, 9, 7]; most of these typically propose a modal weak Kleene semantics based on a *contamination principle*—where the third value infects formulas across world coordinates—and often employ external operators in the resulting logics. Furthermore, it remains unclear whether the characterization theorems of weak Kleene logics, specifically those based on variable inclusion, can be extended to the specific modal versions introduced in these studies, or precisely how those particular systems relate to classical modal logics.

In this work, we begin to fill this gap by providing an internal modal weak Kleene semantics, devoid of external operators. Departing from the contamination principle found in the literature, we draw inspiration from the standard translation theorem of classical modal logic, which translates modal operators into first-order quantifiers. Specifically, we propose a weak Kleene semantics for a modal language based on three-valued Kripke models, in which the interpretation of modal operators remains faithful to a quantificational understanding. Inspired by Malinowski's [8] weak Kleene semantics for first-order quantifiers, we interpret the box operator ($\Box$) as a universal quantification over accessible worlds that behaves like an extended weak Kleene conjunction. Consequently, a modal formula $\Box\varphi$ evaluates to the infectious value (2) at a world w if φ evaluates to 2 at some world accessible from w. This approach not only offers a more intuitive alignment between modal operators and the infectious nature of the third value but also facilitates a translation of this semantics into first-order weak Kleene semantics, effectively extending the standard translation theorem of classical modal logic.

Furthermore, we investigate the paracomplete and paraconsistent weak Kleene logics induced by our semantics. The main technical contribution is a characterization theorem for the modal paracomplete and paraconsistent weak Kleene logics over serial Kripke frames ($\vDash^{\mathbf{ss}}_{\mathbf{D}}$ and $\vDash^{\mathbf{tt}}_{\mathbf{D}}$, respectively). To establish these results, we extend standard techniques from classical Kripke semantics—specifically generated submodels, unraveling, and bisimulation-like invariance—to the weak Kleene framework. We demonstrate that $\Gamma \vDash^{\mathbf{ss}}_{\mathbf{D}} \varphi$ if and only if the inference $\Gamma \vDash_{\mathbf{D}} \varphi$ holds in classical serial modal logic and the variables occurring in φ at any modal depth n are included in the variables occurring in Γ at the same depth n (provided Γ is classically inconsistent). Here, modal depth n is inductively defined as occurrence under the scope of n modal operators. A dual result is provided for $\vDash^{\mathbf{tt}}_{\mathbf{D}}$. These effectively extend the established characterizations of non-modal weak Kleene logics to the modal realm via a refined, modal depth-sensitive variable inclusion constraint. These results also lay the groundwork for a more systematic investigation of modal weak Kleene logics, including characterization theorems for logics defined over different classes of frames.

Finally, we propose philosophical interpretations of our semantics and characterization results that coherently extend those of non-modal weak Kleene logics. In particular, the topic-sensitive interpretation of the third value, as proposed by Beall [**?**], appears promising. If the third value represents *off-topic* content, our characterization suggests that modal operators within our semantics are not topic-transparent. Consequently, a formula's topic is determined not only by its atomic variables but also by the modal depth at which they occur. Thus, our modal weak Kleene logics can be interpreted as preserving *truth-and-on-topicness*, where being *on-topic* requires a structural preservation of variable occurrences across specific modal depths.

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Knowledge on a Budget

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Topological evidence logic (TEL) is a recent branch of epistemic logic that uses topological notions to model epistemic concepts such as justification and knowledge; see [2, 3, 4, 5], for example. In TEL, open sets in a topological space $\langle X, \mathcal{T}\rangle$ represent accessible evidence. A hypothesis $P \subseteq X$ is *justified* if P is supported by a dense open set, i.e. there is evidence $U \in \mathcal{T}$ such that $U \subseteq P$ (U supports P) and $U \cap V \neq \emptyset$ for all $V \in \mathcal{T}$ such that $V \neq \emptyset$ (U is consistent with all consistent evidence). Hypothesis P is known in a state $x \in X$ if it is supported by a dense open neighbourhood of x, that is, there is $U \in \mathcal{T}$ satisfying the conditions given above and, moreover, $x \in U$ (U is truthful).

In practice, accessing evidence requires resources. Real-life agents operate within limited resource budgets, meaning that there is only so much time, money, energy, memory, etc. they can afford to spend on obtaining evidence.

Our work develops a variant of TEL that takes resource limitations into account. We introduce *semiring-annotated topological spaces*, structures that extend topological spaces $\langle X, \mathcal{T}\rangle$ with an annotation function $\mathcal{A}_K$ from $\mathcal{T} \times X$ to the power set of a semiring K; the idea is that $\mathcal{A}_K(U, x)$ is the collection of resources $a \in K$ that are sufficient to access the evidence $U \in \mathcal{T}$ in state $x \in X$. The annotation function satisfies the following conditions:

(1) if $a \in \mathcal{A}_K(U, x)$, then $\{a \cdot b, b \cdot a\} \subseteq \mathcal{A}_K(U, x)$ for all $a, b \in K$;

(2) if $\{a, b\} \subseteq \mathcal{A}_K(U, x)$, then $a + b \in \mathcal{A}_K(U, x)$;

(3) if $a \in \mathcal{A}_K(U, x)$ and $b \in \mathcal{A}_K(V, x)$, then $a \cdot b \in \mathcal{A}_K(U \cap V, x)$;

(4) if $a \in \mathcal{A}_K(U_i, x)$, then $a \in \mathcal{A}_K\big(\bigcup_{i \in I} U_i, x\big)$.

These reflect the reading of $a \cdot b$ as 'a together with b' (resource combination), $a + b$ as 'using a or b' (resource choice), $U \cap V$ as 'evidence U combined with evidence V' and $\bigcup_{i \in I} U_i$ as representing a collection of 'options' $\{U_i\}_{i \in I}$ one can use in support of a hypothesis.

For example, let X be the set of countable binary words (e.g. outputs of a sensor or a program), a directed-compete partially ordered set under the prefix order, and let O be a set of finite binary words containing the empty word (the 'possible observations'). Let $\mathcal{T}_O$ be the topology on X generated by the basis $\{\uparrow w \mid w \in O\}$; open sets in this topology represent 'observable properties' of binary words. Intuitively, the resource spent in obtaining observations w (finite binary words) is *time*, conveniently represented by the length of w. The corresponding resource semiring is extended natural numbers $\mathbb{N}^{\infty}$ with max as semiring multiplication (resource combination).

A K-*frame* is a K-annotated topological space. Given a set of propositional variables *Prop*, a K-*model* is a K-frame $\langle X, \mathcal{T}, \mathcal{A}_K \rangle$ together with a valuation $\mathcal{V} : Prop \to 2^X$. K-models interpret formulas of language $\mathfrak{L}_K$, obtained from *Prop* using $\neg$, $\wedge$, $\Box$ and F_a for $a \in K$. The satisfaction clauses are those familiar from topological semantics of modal logic [1], including $\boldsymbol{M}, x \models \Box\varphi$ iff $\exists U \in \mathcal{T}(x \in U \subseteq [\![\varphi]\!]_{\boldsymbol{M}})$, i.e. x is in the interior of $[\![\varphi]\!]_{\boldsymbol{M}} = \{y \mid \boldsymbol{M}, y \models \varphi\}$, extended with

$$\boldsymbol{M}, x \models \mathrm{F}_a\varphi \iff \exists U \in \mathcal{T}(U \subseteq [\![\varphi]\!]_{\boldsymbol{M}} \ \ \& \ \ a \in \mathcal{A}_K(U, x)).$$

Hence, $\Box\varphi$ means that there is truthful evidence for φ and $\mathrm{F}_a\varphi$ means that evidence for φ can be obtained using resource a. Formulas $\Box_a\varphi$, defined as $\Box\varphi \wedge \mathrm{F}_a\varphi$, say that truthful evidence for φ can be obtained using a.

We provide a sound and weakly complete axiomatisation for the logic of all K-frames and for the logics of several frame classes defined by natural frame conditions, for example (5) $a \in \mathcal{A}_K(U, x)$ and $b \leq a$ only if $b \in \mathcal{A}_K(U, x)$, assuming that K is idempotent (closure under stronger evidence); (6) $1 \in \mathcal{A}_K(X, x)$ (tautologous evidence is for free); (7) $0 \in \mathcal{A}_K(U, x)$ (each open is annotated); and (8) $\mathcal{A}_K(U, x) = \mathcal{A}_K(U, y)$ (uniformity).

Next, we consider the language $\mathfrak{L}_K^{\forall}$ that adds to $\mathfrak{L}_K$ the universal modality $[\forall]$. In $\mathfrak{L}_K^{\forall}$, we can define budget-relative justification and knowledge modalities B_b^a and K_b^a such that

$$\begin{aligned}\boldsymbol{M}, x \models \mathrm{B}_b^a\varphi \iff \exists U \in \mathcal{T}\Big(&U \subseteq [\![\varphi]\!]_{\boldsymbol{M}} \ \ \& \ \ a \in \mathcal{A}_K(U, x) \ \ \& \\ &\forall V \in \mathcal{T}(b \in \mathcal{A}_K(V, x) \ \ \& \ \ V \neq \emptyset \implies U \cap V \neq \emptyset)\Big)\end{aligned}$$

$$\begin{aligned}\boldsymbol{M}, x \models \mathrm{K}_b^a\varphi \iff \exists U \in \mathcal{T}\Big(&x \in U \subseteq [\![\varphi]\!]_{\boldsymbol{M}} \ \ \& \ \ a \in \mathcal{A}_K(U, x) \ \ \& \\ &\forall V \in \mathcal{T}(b \in \mathcal{A}_K(V, x) \ \ \& \ \ V \neq \emptyset \implies U \cap V \neq \emptyset)\Big)\end{aligned}$$

That is, $\mathrm{B}_b^a\varphi$ says that one can obtain evidence U for φ using a such that U is consistent with all consistent evidence V that can be obtained using b. For $\mathrm{K}_b^a\varphi$ to hold, U has to be truthful. If the frame underlying $\boldsymbol{M}$ satisfies conditions (5) to (8) (it is a 'sub-frame'), then B_0^0 and K_0^0 correspond to the justified belief and knowledge operators of [2]. The B_b^a operator, read as a belief operator as in [2], can be used to express that an agent is susceptible to misleading evidence: for a 'small' $\epsilon \in K$, the formula $\mathrm{B}_\epsilon^\epsilon\varphi \wedge \neg\Box\varphi$ means that there is evidence for φ that is cheap for the agent, in that it requires little 'energy' for the agent to accept it, and it is consistent with all other cheap evidence, yet φ is not supported by any truthful evidence (e.g. φ is a hoax).

We provide sound and weakly complete axiomatisations for the $\mathfrak{L}_K^{\forall}$-logics of several K-frame classes, including all frames and all sub-frames.

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When ‘Every S is P’ Became Hypothetical: Rediscovering Herbart

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In classical modern logic, sentences of the form “Every S is P” are analyzed as having the logical form “For every x, if x is S, then x is P.” In older terminology, this means that categorical judgments are treated as hypothetical. My question is why this shift occurred and why categorical judgments came to be understood hypothetically in the first place. At first sight, one might think that the reason lies in the absence of existential import in universal judgments as interpreted in modern logic. But this cannot have been the original motivation, since the lack of existential import created significant tension with Aristotelian logic — a conflict better understood as a *consequence* of the new interpretation. Cohen and Nagel (1934, 42–43) suggest another explanation, namely that treating universal judgments as hypotheses reflects their scientific use. Yet this again seems more a consequence than a historical motive, especially since the shift was not initiated by philosophers of science. In this talk, I focus instead on the historical background of this interpretive change, specifically on developments in nineteenth-century logic and, in particular, on Herbart. Russell in *On Denoting* explicitly appeals to Bradley when he claims that all categorical judgments are hypothetical (Russell 1905, 481). But Bradley’s own discussion repeatedly engages with Herbart (Bradley 1950, 42 ff.). As Sullivan (1991, 142) notes, when Frege eliminates existence from universal judgments, he may also be consciously following Herbart’s lead.

Early in the nineteenth century, Herbart argued that a categorical judgment is never existential but always hypothetical. The reasons for this view reach back to Kant’s critique of the ontological argument. Kant’s insight that existence is not a real predicate was left only half-developed, for he continued to classify existential judgments as categorical — and, moreover, as *synthetic* categorical judgments.

Next, I show how Herbart alters the conception of existential judgments by assigning them an independent status alongside other types of judgments. While Kant claims that in existential judgments there is no predicate and that existence consists merely in the positing of the object of the subject-concept, Herbart insists that existential judgments are “subjectless propositions” expressing the unrestricted positing of a predicate-concept. Existence thus becomes, for the first time, a non-trivial property of concepts.

This reconceptualization has consequences for categorical judgments. If existence is a property of concepts and existential judgments are not structured as subject–predicate combinations, then categorical judgments — which combine concepts — cannot carry existential import. They must therefore be understood hypothetically.

I will argue that Herbart's reworking of the relation between existence and concept provides the philosophical background for the modern interpretation of universal categorical judgments as hypothetical.

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