

On the structure of involutive po-monoids as models of Multiplicative Linear Logic

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Outline

- Involutive po-monoids as models of multiplicative linear logic MLL (CyMLL, RW_m)
- Examples: binary relations, pointed groups, po-groups, Sugihara monoids
- Residuated po-semigroups as models of Lambek Calculus
- The structure of finite ipo-monoids with a Sugihara component

Involutive po-monoids

An **involutive po-monoid** (ipo-monoid) is an algebraic structure $(A, \leq, \cdot, \sim, -, 0)$ such that (A, \leq) is a poset, \cdot is **associative** and

$$(lin) \quad x \leq y \iff x \cdot \sim y \leq 0 \iff -y \cdot x \leq 0$$

Even without associativity (lin) has some interesting consequences:

Lemma

$-, \sim$ form a dual Galois connection.

Proof.

$$-x \leq y \iff -x \cdot \sim y \leq 0 \iff \sim y \leq x. \quad \square$$

Hence $-, \sim$ are order reversing, $-\sim x \leq x$, $\sim -x \leq x$,
 $\sim \sim x = x$ and $-\sim -x = -x$.

But wait, there is more!

The linear negations are involutive

Lemma

$\sim -x = x = -\sim x$, i.e., $\sim, -$ are involutions.

Proof.

$$x \leq -\sim x \iff x \cdot \sim -\sim x \leq 0 \iff x \cdot \sim x \leq 0 \iff x \leq x. \quad \square$$

For the next few results we use associativity.

Lemma

$\sim 0 = -0$ is an identity element.

Proof.

$-0 \cdot x \leq y \iff -0 \cdot x \cdot \sim y \leq 0 \iff x \cdot \sim y \leq 0 \iff x \leq y$. Hence $-0 \cdot x = x$ and similarly $x \cdot \sim 0 = x$. Therefore $\sim 0 = -0 \cdot \sim 0 = -0$. \square

Naturally ~ 0 is denoted by 1.

Multiplication is residuated, hence order-preserving

Lemma

$$xy \leq z \iff x \leq -(y \cdot \sim z) \iff y \leq \sim(-z \cdot x).$$

Proof.

$$xy \leq z \iff (-\sim x)y \sim z \leq 0 \iff y \sim z \leq \sim x \iff x \leq -(y \sim z).$$

Similarly

$$xy \leq z \iff -zx(\sim -y) \leq 0 \iff -zx \leq -y \iff y \leq \sim(-zx). \quad \square$$

Therefore $z/y = -(y \cdot \sim z)$ and $x \setminus z = \sim(-z \cdot x)$.

Lemma

$$x \leq y \implies xz \leq yz \text{ and } zx \leq zy.$$

Proof.

$$\text{Follows from residuation: } yz \leq yz \implies x \leq y \leq yz/z \implies xz \leq yz. \quad \square$$

Algebraic models of multiplicative linear logic (MLL)

The previous results show that ipo-monoids are po-algebraic models of (non-commutative, non-cyclic) **multiplicative linear logic** (MLL). MLL entailment has FMP and is decidable (NP-complete!).

With cyclicity ($xy \leq 0 \Leftrightarrow yx \leq 0$) added, we obtain models of **cyclic multiplicative linear logic** (CyMLL), where $\sim x = -x$.

With commutativity ($xy \leq yx$) added, we obtain models of **multiplicative fragment of contraction-free relevance logic** (RW_m^t).

We will see the **finite ipo-monoids** come in a form of **group**-indexed unions of bounded connected components $\bigcup_{g \in G} A_g$. Depending on additional axioms or assumptions on A_e , we can say a little bit more.

Examples: binary relations, pointed groups, po-groups

Any set of binary relations $A \subseteq \mathcal{P}(X^2)$ closed under composition and **complement-inverse** $\sim R = -R = X^2 \setminus R^{-1}$.

A **pointed group** $(A, \cdot, {}^{-1}, 1, 0)$ is a group with an (arbitrary) constant 0.

To obtain an ipo-monoid, let \leq be equality and define $\sim x = x^{-1} \cdot 0$ and $-x = 0 \cdot x^{-1}$.

A **partially ordered group** is a structure $\mathbf{A} = (A, \leq, \cdot, {}^{-1}, 1)$ such that

- 1 (A, \leq) is a poset
- 2 $(A, \cdot, {}^{-1}, 1)$ is a group and
- 3 multiplication is order preserving (hence ${}^{-1}$ is order reversing).

$\implies 0 = 1^{-1} = 1$ and **cyclicity** holds: $\sim x = -x = x^{-1}$.

Examples: $(\mathbb{R}, \leq, +, -, 0)$, $(\mathbb{Q}^+, \leq, \cdot, {}^{-1}, 1)$, **all** groups (with \leq is $=$)

Sugihara ipo-monoids (algebraic models of RM_m)

S_{2n} : a $2n$ -element chain $a_n < \dots < a_2 < \mathbf{0} < 1 < b_2 < \dots < b_n$

S_{2n-1} : a $2n-1$ -element chain $a_n < \dots < a_2 < \mathbf{0=1} < b_2 < \dots < b_n$.

Define $a_1 = 0$, $b_1 = 1$, $\sim a_i = -a_i = b_i$, $\sim b_i = -b_i = a_i$,

$$a_i \cdot a_j = a_{\max(i,j)}, \quad b_i \cdot b_j = b_{\max(i,j)}, \quad a_i \cdot b_j = b_j \cdot a_i = \begin{cases} b_i & \text{if } i < j \\ a_j & \text{otherwise} \end{cases}$$

$(S_m, \leq, \cdot, \sim, -, 0)$ is the m -element **Sugihara ipo-monoid** (no \wedge, \vee)

$S_2 = \mathbf{2}$ the two-element **Boolean algebra**

S_3 is also known as the **Gaifman-Pratt ipo-monoid**

Models from products of groups and ipo-chains

Let G be a group with subgroup H .

Define $A = H \times S_2 \cup (G \setminus H) \times S_1$ with

$$(g, a) \cdot (g', a') = (gg', aa') \text{ and } \sim(g, a) = (g^{-1}, \sim a) = -(g, a).$$

Then A is an ipo-monoid.

Our structure theorem says that any finite ipo-monoid where the connected component with 1 is isomorphic to S_m will be a **union of chains of length $\leq m$** with its monoid structure uniquely determined by a sequence of **nested subgroups of G** .

This is a generalization of closely related results by Zhuang [2023], Galatos and Zhuang [2024] on **unilinear residuated lattices**.

Partially ordered algebras

A **po-algebra** is a partially ordered set with operations that are either order-preserving or **order-reversing** in each argument.

A **variety of po-algebras** is a class of similar po-algebras defined by equations or **inequations** [Pigozzi 2004].

A **residuated partially ordered magma** or **rpo-magma**

$\mathbf{A} = (A, \leq, \cdot, \backslash, /)$ is a partially-ordered set (A, \leq) with a binary operation \cdot and two **residuals** that satisfy for all $x, y, z \in A$

$$\text{(res)} \quad xy \leq z \iff x \leq z/y \iff y \leq x \backslash z$$

The operation $x \cdot y$ is usually written xy .


Residuated po-magmas are a po-variety

Residuation ensures that x/y and $y \setminus x$ are order-preserving in the numerator (x position) and **order-reversing** in the denominator.

xy is order-preserving in both arguments.

(res) is equivalent to $x \leq xy/y$, $(z/y)y \leq z$, $y \leq x \setminus xy$, $x(x \setminus z) \leq z$ hence rpo-magmas are a variety of po-algebras.

Although rpo-magmas are very general, (res) imposes restrictions on the posets that can occur.

E.g. could  be the poset of a rpo-magma?

Lemma

For rpo-magmas, if $a, b \leq c$ then $(a/(a \setminus c))((c/(a \setminus c)) \setminus b) \leq a, b$.

Proof.

Assume $a \leq c$.	Then	$a/(a \setminus c) \leq c/(a \setminus c)$
\implies		$(c/(a \setminus c)) \setminus b \leq (a/(a \setminus c)) \setminus b$
\iff		$(a/(a \setminus c))((c/(a \setminus c)) \setminus b) \leq b$
Assume $b \leq c$.	Then	$a \setminus c \leq a \setminus b$
\iff		$a(a \setminus c) \leq a \setminus b$
\iff		$a \leq c/(a \setminus c)$
\implies		$(c/(a \setminus c)) \setminus b \leq a \setminus b \leq a \setminus c$
\implies		$a/(a \setminus c) \leq a/((c/(a \setminus c)) \setminus b)$
\iff		$(a/(a \setminus c))((c/(a \setminus c)) \setminus b) \leq a$

□

Finite rpo-magmas have bounded components

Lemma

In any rpo-magma, if $d \leq a, b$ then

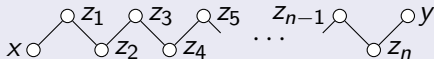
$$a, b \leq d / (((a \setminus d) / (a \setminus (d \setminus d)))((d \setminus d) / (a \setminus (d \setminus d))) \setminus (b \setminus d))$$

Theorem

In an rpo-magma every connected component of \leq is up-directed and down-directed, hence for **finite** rpo-magmas every connected component is **bounded**.

Proof.

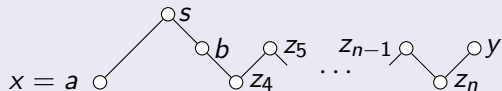
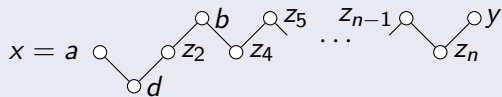
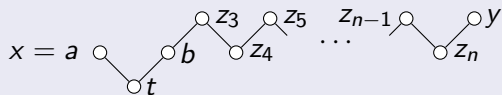
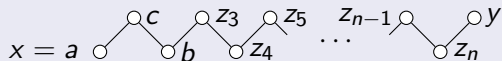
Two elements x, y in a poset are connected iff there exists a zigzag



We need to find an upper and a lower bound of x, y .



Proof (continued).



Use induction to get an upper and lower bound of x, y in n steps.

What posets are possible for rpo-magmas?

In a poset define $x \rightsquigarrow y$ if x and y are in the same connected component.

Theorem

For any po-algebra the relation \rightsquigarrow is a congruence.

*For a rpo-magma A the quotient algebra A/\rightsquigarrow is a **quasigroup** with \leq as equality, i.e., $xy = z \iff x = z/y \iff y = x \setminus z$.*

*Conversely, from any quasigroup Q and a pairwise disjoint family of **bounded** posets A_q for $q \in Q$, one can construct an rpo-magma with poset $\bigcup_{q \in Q} A_q$.*

E.g. for a quasigroup Q and $x \in A_p, y \in A_q$ define

$$x \cdot y = \perp_{pq}, \quad x \setminus y = \top_{p \setminus q}, \quad x / y = \top_{p/q}.$$

Residuated po-semigroups

A **rpo-semigroup** or **Lambek algebra** is a rpo-magma where \cdot is associative. These are algebraic models of Lambek Calculus.

Lemma

*If the poset is an antichain then a rpo-semigroup is a **group**.*

Proof.

An associative quasigroup is a group: $x = y \backslash yx = y(y \backslash x)$. Hence $x = x(x \backslash x) \Rightarrow xy = x(x \backslash x)y \Rightarrow y = x \backslash xy = (x \backslash x)y \Rightarrow y / y = x \backslash x$. So $x \backslash x = e$ is constant, $xe = x$ and $ey = y$, i.e., e is an identity.

Now $(e/x)x = e \Rightarrow x(e/x)x = x \Rightarrow x(e/x) = x/x = e$, so $x^{-1} = e/x$. \square

\Rightarrow For any rpo-semigroup A the quotient A/\sim is a group.

Ipo-monoids with additional assumptions

For an ipo-monoid (rpo-monoid) A , the connected components are denoted C_g for $g \in A/\sim$. The identity component C_e contains the identity element 1.

The **order** of $x \in A$ is the smallest positive n such that $x^n \in C_e$, or ∞ if no finite n exists.

In finite ipo-monoids, every element has a finite order, and they all divide the order of the governing group, so $\exists n \forall x (x^n \sim 1)$. We are interested in properties of the map $f : x \mapsto x^n$.

- Ipo-monoids where $0 \leq 1$
- Ipo-monoids with $x^{n+1} = x$ for some n (periodic n -contraction)
- Ipo-monoids where $x \leq 0$ or $1 \leq x$ for all $x \in C_e$
- Ipo-monoids where $C_e = S_m$

Ipo-monoids where $0 \leq 1$

Lemma

In an ipo-monoid with $0 \leq 1$, for any n , $(\sim x)^n \leq \sim x^n$ and $(-x)^n \leq -x^n$.

Proof.

By induction, assume $(\sim x)^n \leq \sim x^n$. Then $x^n(\sim x)^n \leq 0 \leq 1$.

Now $x \cdot x^n(\sim x)^n \cdot \sim x \leq x \cdot \sim x \leq 0$.

Hence $x^{n+1}(\sim x)^{n+1} \leq 0$ and therefore $(\sim x)^{n+1} \leq \sim x^{n+1}$. □

Ipo-monoids where $x \leq 0$ or $1 \leq x$ for all $x \in C_e$

Note that $x \leq 0$ or $1 \leq x$ holds in S_m .

Lemma

Let A be an ipo-monoid such that $0 \leq 1$ and $(x \leq 0 \text{ or } 1 \leq x)$ for all $x \in C_e$. If $x, y \in C_g$ have order n then $f : C_g \rightarrow C_e$ given by $f(x) = x^n$ is injective.

Proof.

Assume $x^n \leq y^n$. We want to show $x \leq y$, then injectivity follows.

Suppose $x \not\leq y$, then $x \cdot \sim y \not\leq 0$. Note $x, y \in C_g \implies x \cdot \sim y \in C_e$.

From $(z \leq 0 \text{ or } 1 \leq z)$ for $z \in C_e$ we conclude $1 \leq x \cdot \sim y$.

Hence $1 \leq x \cdot \sim y \leq x \cdot 1 \cdot \sim y \leq x \cdot x \cdot \sim y \cdot \sim y \leq \dots \leq x^n \cdot (\sim y)^n$.

By the preceding lemma and $x^n \leq y^n$, $x^n \cdot (\sim y)^n \leq x^n \cdot \sim y^n \leq 0$.

Therefore $x \cdot \sim y \leq 0$, a contradiction. □

Structure of finite ipo-monoids with $C_e \cong S_m$

Theorem

Let A be a finite ipo-monoid with $C_e \cong S_m$. Then A is a disjoint union of chains, each with $\leq m$ elements, and \cdot determined by subgroups of A/\sim

Proof (outline).

Let $G = A/\sim$. The map $f : C_g \rightarrow C_e$ is an **order embedding** since $g(x) = x^{n+1}$ is injective on the finite set C_g , hence surjective.

Therefore all connected components are chains with $\leq m$ elements.

For $1 \leq k \leq m$, let $H_k = \{g \in G \mid k \leq |C_g|\}$.

Then $H_m \leq \dots \leq H_2 \leq H_1 = G$ is a nested sequence of subgroup of G and these subgroups uniquely determine the multiplication on the algebra. \square

These algebras can also be constructed using Płonka sums over a linear join-semilattice direct system from ipo-monoids with $C_e = S_2$ and, if $0 = 1$, the least summand has $C_e = S_1$, i.e., it is a group.

Ipo-monoids with $x^{n+1} = x$ and $0 < 1$

Note that **periodic n-contraction** $x^{n+1} = x$ is weaker than **periodicity** $x^n = 1$ considered in the context of residuated lattices. It holds for ipo-monoids with $C_e \cong S_m$.

- $x^{n+1} = x$ means $x \sqsubseteq x^n$ (the monoid-induced order)
- $x^{n+1} = x$ implies $x^n \approx 1$, and x^n is idempotent, and $(x^n)^n = x^n$.
- $x^{n+1} = x$ IFF $x \cdot \sim x \leq x^n \leq x \setminus x$ and $-x \cdot x \leq x^n \leq x / x$

Lemma

In a finite ipo-monoid A satisfying $x^{n+1} = x$ for $n = |G|$, and $0 < 1$, with $f : x \mapsto x^n$:

- *Fixpoints of f coincide with the idempotents of A , and they form a cyclic balanced subalgebra of A (in fact of C_e)*
- *$A' = \{x \mid x \leq 1 \text{ or } 1 \leq x\}$ is a subalgebra of $\text{Idmp}(A)$, where $\{x \mid 0 \leq x \leq 1\}$ is a Boolean algebra, and $\{x \mid x \leq 0 \text{ or } 1 \leq x\}$ is a Sugihara monoid.*
- *Therefore $\{x \mid x \leq 1\}$ is a Heyting algebra.*

Thank You!








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