Termination, Countermodel and Complexity in Bilateral Labeled Sequent Calculi

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1. Connexivity and Bilateralism
2. Proof Theory & Completeness
3. Termination, Countermodel & Complexity
4. Conclusion

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1. Connexivity and Bilateralism
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Connexivity and connexive logics

- Contra-classical logics: systems that are neither subsystems nor extensions of classical logic.
- Connexivity is related to a specific set of formulas:
  - Aristotle's Theses (AT): $\sim(\sim A \to A)$ and $\sim(A \to \sim A)$;
  - Boethius' Theses (BT): $(A \to B) \to \sim(A \to \sim B)$ and $(A \to \sim B) \to \sim(A \to B)$;
  - Non-symmetric implication: (Sym) $\nvdash (A \to B) \to (B \to A)$.

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Variety of logics and aims

1. R. Angell's system formalizing a concept 'contrariety'.
2. S. McCall introduced the term connexive logic; Represent all valid moods of Aristotelian syllogistic.
3. C. Pizzi: logics of consequential implication.
4. R. Sylvan, R. Brady, C. Mortensen: connexive relevant logic.
5. N. Francez: poly-connexivity (conjunction and disjunction).
6. . . .

Constructive connexive logic

- Nelson's constructive four-valued logic with strong negation N4.
- N4 extends FDE with intuitionistic implication.

N4 (axioms and rules)

$$A \to (B \to A) \quad \text{(Ax1)}$$
$$(A \to B) \to ((A \to (B \to C)) \to (A \to C)) \quad \text{(Ax2)}$$
$$A \to (B \to (A \wedge B)) \quad \text{(Ax3)}$$
$$(A \wedge B) \to A \quad \text{(Ax4)}$$
$$(A \wedge B) \to B \quad \text{(Ax5)}$$
$$A \to (A \vee B) \quad \text{(Ax6)}$$
$$B \to (A \vee B) \quad \text{(Ax7)}$$
$$(A \to C) \to ((B \to C) \to ((A \vee B) \to C)) \quad \text{(Ax8)}$$
$$\sim\sim A \leftrightarrow A \quad \text{(Ax9)}$$
$$\sim(A \wedge B) \leftrightarrow (\sim A \vee \sim B) \quad \text{(Ax10)}$$
$$\sim(A \vee B) \leftrightarrow (\sim A \wedge \sim B) \quad \text{(Ax11)}$$
$$\sim(A \to B) \leftrightarrow (A \wedge \sim B) \quad \text{(Ax12)}$$
$$A, A \to B \Rrightarrow B \quad \text{(MP)}$$
$$A, B \Rrightarrow A \wedge B \quad \text{(R}\wedge\text{)}$$

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N4 (axioms and rules)

$$
\begin{array}{lr}
A \to (B \to A) & (\text{AX1}) \\
(A \to B) \to ((A \to (B \to C)) \to (A \to C)) & (\text{AX2}) \\
A \to (B \to (A \wedge B)) & (\text{AX3}) \\
(A \wedge B) \to A & (\text{AX4}) \\
(A \wedge B) \to B & (\text{AX5}) \\
A \to (A \vee B) & (\text{AX6}) \\
B \to (A \vee B) & (\text{AX7}) \\
(A \to C) \to ((B \to C) \to ((A \vee B) \to C)) & (\text{AX8}) \\
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\sim(A \to B) \leftrightarrow (A \wedge \sim B) & (\text{AX12}) \\
A, A \to B \Rrightarrow B & (\text{MP}) \\
A, B \Rrightarrow A \wedge B & (\text{R}\wedge)
\end{array}
$$

- Replace (AX12) by $\sim(A \to B) \leftrightarrow (A \to \sim B)$ (BT) to get the constructive connexive logic C.

Bilateralism and the semantics of C

- Bilateralism: clauses or rules for both asserting/proving and denying/refuting.
- Wansing, H. (2005). "Connexive modal logic", In: *AiML*, vol. 5, 387–399.

Bilateralism and the semantics of C

- Bilateralism: clauses or rules for both asserting/proving and denying/refuting.
- Wansing, H. (2005). "Connexive modal logic", In: *AiML*, vol. 5, 387–399.
- Semantics operates over information states pre-ordered by possible expansion relation.

Definition. (C-frame)

A C-frame, denoted $\mathscr{F}$, is a structure of the following shape $\langle W, \leqslant \rangle$, where W is a set of points (worlds, states) and $\leqslant \subseteq W^2$, satisfying the following conditions: (1) $\forall x \in W \, (x \leqslant x)$ and (2) $\forall x, y, z \in W \, (x \leqslant y \wedge y \leqslant z \implies x \leqslant z)$.

- No unilateral satisfaction relation $\vDash$.
- Bilateral satisfaction relations: support of truth ($\vDash^+$) and support of falsity ($\vDash^-$).

Connectives in the bilateral semantics

Definition. (C-model)

A C-model is a structure $\mathscr{M} = \langle \mathscr{F}, \mathsf{v}^+, \mathsf{v}^- \rangle$, where $\mathscr{F}$ is a C-frame and v^+ and v^- are valuation functions from At into $\langle W, \leqslant \rangle$ s.t., for every $p \in$ At, if $x \in \mathsf{v}^\bullet(p)$ and $x \leqslant y$, then $y \in \mathsf{v}^\bullet(p)$, for $\bullet \in \{+, -\}$. The relations $\mathscr{M}, x \vDash^+ A$ and $\mathscr{M}, x \vDash^- A$ are defined as follows:

$\mathscr{M}, x \vDash^+ p$ iff $x \in \mathsf{v}^+(p)$
$\mathscr{M}, x \vDash^- p$ iff $x \in \mathsf{v}^-(p)$

$\mathscr{M}, x \vDash^+ {\sim}A$ iff $\mathscr{M}, x \vDash^- A$
$\mathscr{M}, x \vDash^- {\sim}A$ iff $\mathscr{M}, x \vDash^+ A$

$\mathscr{M}, x \vDash^+ A \wedge B$ iff $\mathscr{M}, x \vDash^+ A$ and $\mathscr{M}, x \vDash^+ B$
$\mathscr{M}, x \vDash^- A \wedge B$ iff $\mathscr{M}, x \vDash^- A$ or $\mathscr{M}, x \vDash^- B$

$\mathscr{M}, x \vDash^+ A \vee B$ iff $\mathscr{M}, x \vDash^+ A$ or $\mathscr{M}, x \vDash^+ B$
$\mathscr{M}, x \vDash^- A \vee B$ iff $\mathscr{M}, x \vDash^- A$ and $\mathscr{M}, x \vDash^- B$

$\mathscr{M}, x \vDash^+ A \rightarrow B$ iff $\forall y \in W : x \leqslant y$ and $\mathscr{M}, y \vDash^+ A$, imply $\mathscr{M}, y \vDash^+ B$
$\mathscr{M}, x \vDash^- A \rightarrow B$ iff $\forall y \in W : x \leqslant y$ and $\mathscr{M}, y \vDash^+ A$, imply $\mathscr{M}, y \vDash^- B$

If $\mathscr{M} = \langle W, \leqslant, \mathsf{v}^+, \mathsf{v}^- \rangle$ is a C-model, then $\mathscr{M} \vDash A$ iff for every $x \in W$, $\mathscr{M}, x \vDash^+ A$. Moreover, $\mathscr{F} \vDash A$ holds iff $\mathscr{M} \vDash A$ for every model $\mathscr{M}$ based on $\mathscr{F}$. Finally, a formula is C-valid iff it is valid on every C-frame.

Results from Wansing (2005)

Proposition. (Monotonicity)

If $x \models^{\bullet} A$ and $x \leqslant y$, then $y \models^{\bullet} A$, for $\bullet \in \{+,-\}$, $A \in$ Frm and $x, y \in W$.

Theorem. (Soundness & Completeness)

A is a C-theorem iff A is C-valid.

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Bilateral Labeled Sequents

- Gentzen's sequents: $\Gamma \Rightarrow \Delta$.
- Labeled sequents: enrichment of the syntax of sequents via labels $x, y, \ldots$.
- Labeled formulas $x : A$ and relational atoms $x \leqslant y$.

Bilateral Labeled Sequents

- Gentzen's sequents: $\Gamma \Rightarrow \Delta$.
- Labeled sequents: enrichment of the syntax of sequents via labels $x, y, \ldots$.
- Labeled formulas $x : A$ and relational atoms $x \leqslant y$.
- Bilateral labeled sequents: further enrichment of the syntax via $+$ and $-$.
- Bilateral labeled formulas $x :^{+} A$ and $x :^{-} A$, and relational atoms $x \leqslant y$.
- Γ multiset of labeled formulas and relational atoms; Δ multiset of labeled formulas.

Example.

Let $\Gamma = y \leqslant y, x \leqslant z, :^{+} p, y :^{-} p \rightarrow q, y :^{-} q$ and $\Delta = z :^{+} p, y :^{+} q \wedge p, z :^{-} q$.
Then, $\Gamma \Rightarrow \Delta$ is the following sequent:

$$y \leqslant y, x \leqslant z, x :^{+} p, y :^{-} p \rightarrow q, y :^{-} q \Rightarrow z :^{+} p, y :^{+} q \wedge p, z :^{-} q$$

Bilateral Labeled Sequent Calculus GC^{+}_{-}

Verification rules.

INITIAL SEQUENTS. $x \leqslant y, x :^{+} p, \Gamma \Rightarrow \Delta, y :^{+} p$

RULES FOR $\sim$, $\wedge$, $\vee$ AND $\rightarrow$.

$$\frac{\Gamma \Rightarrow \Delta, x :^{-} A}{\Gamma \Rightarrow \Delta, x :^{+} {\sim} A}\ R^{+}_{\sim} \qquad \frac{x :^{-} A, \Gamma \Rightarrow \Delta}{x :^{+} {\sim} A, \Gamma \Rightarrow \Delta}\ L^{+}_{\sim}$$

$$\frac{\Gamma \Rightarrow \Delta, x :^{+} A \qquad \Gamma \Rightarrow \Delta, x :^{+} B}{\Gamma \Rightarrow \Delta, x :^{+} A \wedge B}\ R^{+}_{\wedge} \qquad \frac{x :^{+} A, x :^{+} B, \Gamma \Rightarrow \Delta}{x :^{+} A \wedge B, \Gamma \Rightarrow \Delta}\ L^{+}_{\wedge}$$

$$\frac{\Gamma \Rightarrow \Delta, x :^{+} A, x :^{+} B}{\Gamma \Rightarrow \Delta, x :^{+} A \vee B}\ R^{+}_{\vee} \qquad \frac{x :^{+} A, \Gamma \Rightarrow \Delta \qquad x :^{+} B, \Gamma \Rightarrow \Delta}{x :^{+} A \vee B, \Gamma \Rightarrow \Delta}\ L^{+}_{\vee}$$

$$(y \text{ fresh})\ \frac{x \leqslant y, y :^{+} A, \Gamma \Rightarrow \Delta, y :^{+} B}{\Gamma \Rightarrow \Delta, x :^{+} A \rightarrow B}\ R^{+}_{\rightarrow} \qquad \frac{x \leqslant y, x :^{+} A \rightarrow B, \Gamma \Rightarrow \Delta, y :^{+} A \qquad y :^{+} B, x \leqslant y, x :^{+} A \rightarrow B, \Gamma \Rightarrow \Delta}{x \leqslant y, x :^{+} A \rightarrow B, \Gamma \Rightarrow \Delta}\ L^{+}_{\rightarrow}$$

Bilateral Labeled Sequent Calculus GC^{+}_{-}

Falsification rules.

INITIAL SEQUENTS. $x \leqslant y, x :^{-} p, \Gamma \Rightarrow \Delta, y :^{-} p$

RULES FOR $\sim$, $\wedge$, $\vee$ AND $\rightarrow$.

$$\frac{\Gamma \Rightarrow \Delta, x :^{+} A}{\Gamma \Rightarrow \Delta, x :^{-} \sim A} \, R^{-}_{\sim} \qquad \frac{x :^{+} A, \Gamma \Rightarrow \Delta}{x :^{-} \sim A, \Gamma \Rightarrow \Delta} \, L^{-}_{\sim}$$

$$\frac{\Gamma \Rightarrow \Delta, x :^{-} A, x :^{-} B}{\Gamma \Rightarrow \Delta, x :^{-} A \wedge B} \, R^{-}_{\wedge} \qquad \frac{x :^{-} A, \Gamma \Rightarrow \Delta \quad x :^{-} B, \Gamma \Rightarrow \Delta}{x :^{-} A \wedge B, \Gamma \Rightarrow \Delta} \, L^{-}_{\wedge}$$

$$\frac{\Gamma \Rightarrow \Delta, x :^{-} A \quad \Gamma \Rightarrow \Delta, x :^{-} B}{\Gamma \Rightarrow \Delta, x :^{-} A \vee B} \, R^{-}_{\vee} \qquad \frac{x :^{-} A, x :^{-} B, \Gamma \Rightarrow \Delta}{x :^{-} A \vee B, \Gamma \Rightarrow \Delta} \, L^{-}_{\vee}$$

$$(y \text{ fresh}) \, \frac{x \leqslant y, y :^{+} A, \Gamma \Rightarrow \Delta, y :^{-} B}{\Gamma \Rightarrow \Delta, x :^{-} A \rightarrow B} \, R^{-}_{\rightarrow} \qquad \frac{x \leqslant y, x :^{-} A \rightarrow B, \Gamma \Rightarrow \Delta, y :^{+} A \quad y :^{-} B, x \leqslant y, x :^{-} A \rightarrow B, \Gamma \Rightarrow \Delta}{x \leqslant y, x :^{-} A \rightarrow B, \Gamma \Rightarrow \Delta} \, L^{-}_{\rightarrow}$$

Relational rules and preliminary results

- Reflexivity and transitivity of $\leqslant$. Add REF and TRS to GC^{+}_{-}:

$$\frac{x \leqslant x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \textsc{ref} \qquad \frac{x \leqslant z, x \leqslant y, y \leqslant z, \Gamma \Rightarrow \Delta}{x \leqslant y, y \leqslant z, \Gamma \Rightarrow \Delta}\ \textsc{trs}$$

Relational rules and preliminary results

- Reflexivity and transitivity of $\leqslant$. Add REF and TRS to GC^{+}_{-}:

$$\frac{x \leqslant x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathrm{REF} \qquad \frac{x \leqslant z, x \leqslant y, y \leqslant z, \Gamma \Rightarrow \Delta}{x \leqslant y, y \leqslant z, \Gamma \Rightarrow \Delta}\ \mathrm{TRS}$$

- Derivability of initial sequents for compound formulas:

$$x \leqslant y, x :^{+} A, \Gamma \Rightarrow \Delta, y :^{+} A \qquad (\mathrm{id}^{+})$$

$$x \leqslant y, x :^{-} A, \Gamma \Rightarrow \Delta, y :^{-} A \qquad (\mathrm{id}^{-})$$

- Monotonicity rules for atomic formulas (Height-preserving admissible):

$$\frac{x \leqslant y, x :^{\bullet} p, y :^{\bullet} p, \Gamma \Rightarrow \Delta}{x \leqslant y, x :^{\bullet} p, \Gamma \Rightarrow \Delta}\ \text{L-MON}_{at} \qquad \frac{x \leqslant y, \Gamma \Rightarrow \Delta, y :^{\bullet} p, x :^{\bullet} p}{x \leqslant y, \Gamma \Rightarrow \Delta, y :^{\bullet} p}\ \text{R-MON}_{at}$$

- Monotonicity rules for compound formulas (admissible):

$$\frac{x \leqslant y, x :^{\bullet} A, y :^{\bullet} A, \Gamma \Rightarrow \Delta}{x \leqslant y, x :^{\bullet} A, \Gamma \Rightarrow \Delta}\ \text{L-MON} \qquad \frac{x \leqslant y, \Gamma \Rightarrow \Delta, y :^{\bullet} A, x :^{\bullet} A}{x \leqslant y, \Gamma \Rightarrow \Delta, y :^{\bullet} A}\ \text{R-MON}$$

Admissibility and invertibility results

Lemma. (Structural rules)

The rules of label substitution, weakening and contraction are height-preserving admissible in GC^+_-. Let φ be either of the form $x :^\bullet A$ or $x \leqslant y$.

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma(y/x) \Rightarrow \Delta(y/x)}\ \textsc{sub}(y/x) \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, x :^\bullet A}\ \textsc{rw} \qquad \frac{\Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\ \textsc{lw}$$

$$\frac{\Gamma \Rightarrow \Delta, x :^\bullet A, x :^\bullet A}{\Gamma \Rightarrow \Delta, x :^\bullet A}\ \textsc{rc} \qquad \frac{\varphi, \varphi, \Gamma \Rightarrow \Delta}{\varphi, \Gamma \Rightarrow \Delta}\ \textsc{lc}$$

Lemma. (Logical rules)

All verification, falsification and relational rules of GC^+_- are height-preserving invertible.

cut-admissibility

cut-admissibility

The rules cut^+ and cut^- are admissible.

$$\frac{\Gamma \Rightarrow \Delta, x :^+ A \qquad x :^+ A, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}\ \mathsf{cut}^+ \qquad \frac{\Gamma \Rightarrow \Delta, x :^- A \qquad x :^- A, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}\ \mathsf{cut}^-$$

cut-admissibility

cut-admissibility

The rules cut^{+} and cut^{-} are admissible.

$$\dfrac{\Gamma \Rightarrow \Delta, x :^{+} A \qquad x :^{+} A, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}\ \mathsf{cut}^{+} \qquad \dfrac{\Gamma \Rightarrow \Delta, x :^{-} A \qquad x :^{-} A, \Pi \Rightarrow \Sigma}{\Gamma, \Pi \Rightarrow \Delta, \Sigma}\ \mathsf{cut}^{-}$$

Assume that the premises of cut^{-} are derived by $R^{-}_{\rightarrow}$ and $L^{-}_{\rightarrow}$, respectively.

$$\dfrac{\dfrac{x \leqslant y, y :^{+} B, \Gamma \Rightarrow \Delta, y :^{-} C}{\Gamma \Rightarrow \Delta, x :^{-} B \rightarrow C}\ R^{-}_{\rightarrow} \qquad \dfrac{x \leqslant z, x :^{-} B \rightarrow C, \Pi \Rightarrow \Sigma, z :^{+} B \qquad x \leqslant z, z :^{-} C, x :^{-} B \rightarrow C, \Pi \Rightarrow \Sigma}{x \leqslant z, x :^{-} B \rightarrow C, \Pi \Rightarrow \Sigma}\ L^{-}_{\rightarrow}}{x \leqslant z, \Gamma, \Pi \Rightarrow \Delta, \Sigma}\ \mathsf{cut}^{-}$$

$\rightsquigarrow$

$$\dfrac{\overline{\overline{\dfrac{\dfrac{\Gamma \Rightarrow \Delta, x :^{-} B \rightarrow C \qquad x \leqslant z, x :^{-} B \rightarrow C, \Pi \Rightarrow \Sigma, z :^{+} B}{x \leqslant z, \Gamma, \Pi \Rightarrow \Delta, \Sigma, z :^{+} B}\ \mathsf{cut}^{-} \qquad \dfrac{x \leqslant y, y :^{+} B, \Gamma \Rightarrow \Delta, y :^{-} C}{x \leqslant z, z :^{+} B, \Gamma \Rightarrow \Delta, z :^{-} C}\ \textsc{sub}(z/y)}{x \leqslant z, x \leqslant z, \Gamma, \Gamma, \Pi \Rightarrow \Delta, \Delta, \Sigma, z :^{-} C}\ \mathsf{cut}^{+}}}\ \textsc{lc+rc}}{x \leqslant z, \Gamma, \Pi \Rightarrow \Delta, \Sigma, z :^{-} C}$$

$$\dfrac{x \leqslant z, \Gamma, \Pi \Rightarrow \Delta, \Sigma, z :^{-} C \qquad \dfrac{\Gamma \Rightarrow \Delta, x :^{-} B \rightarrow C \qquad x \leqslant z, z :^{-} C, x :^{-} B \rightarrow C, \Pi \Rightarrow \Sigma}{x \leqslant z, z :^{-} C, \Gamma, \Pi \Rightarrow \Delta, \Sigma}\ \mathsf{cut}^{-}}{\overline{\overline{x \leqslant z, x \leqslant z, \Gamma, \Gamma, \Pi, \Pi \Rightarrow \Delta, \Delta, \Sigma, \Sigma}}\ \textsc{lc + rc}}\ \mathsf{cut}^{-}$$

$$x \leqslant z, \Gamma, \Pi \Rightarrow \Delta, \Sigma$$

Soundness

Definition. (Satisfaction; Validity)

Let $\mathscr{F} = \langle W, \leqslant \rangle$ be a C-frame and $\bullet \in \{+, -\}$.

- A labeled sequent $\Lambda \equiv \Gamma \Rightarrow \Delta$ is satisfied in a model $\mathscr{M} = \langle \mathscr{F}, \mathsf{v}^+, \mathsf{v}^- \rangle$ under an interpretation $[\![\cdot]\!] : \mathsf{Lab} \longrightarrow W$ if:
  - If $x \leqslant y \in \Gamma$, then $[\![x]\!] \leqslant [\![y]\!]$;
  - If $x :^\bullet A \in \Gamma$, then $\mathscr{M}, [\![x]\!] \vDash^\bullet A$;
  - If $x :^\bullet A \in \Delta$, then $\mathscr{M}, [\![x]\!] \nvDash^\bullet A$.
- Λ is valid on a C-frame $\mathscr{F}$ if it is satisfied in every model $\mathscr{M}$ based on $\mathscr{F}$.
- Λ is C-valid if it is valid on every C-frame.

Theorem. (Soundness)

If $\vdash_{\mathsf{GC}^+_-} \Gamma \Rightarrow \Delta$, then $\Gamma \Rightarrow \Delta$ is C-valid.

Completeness

Theorem.

Let $\Gamma \Rightarrow \Delta$ be a GC^{+}_{-} sequent. Then, either the sequent is derivable in GC^{+}_{-} or it has a countermodel in C-frames $\mathscr{F}$.

Proof strategy (Sketch)

- Construct a derivation tree for $\Gamma \Rightarrow \Delta$ by applying the rules of GC^{+}_{-} root-first.
- If the tree is finite (all leaves are initial sequents), we obtain a proof in GC^{+}_{-}.
- If the tree is infinite, by König's Lemma, there exists an infinite branch.
- From such a branch, build a countermodel making all labelled formulas and relational atoms in Γ true and all labelled formulas in Δ false.
- Define a frame $\mathscr{F}$ on the labels occurring in Γ within the branch.
- Show that, for every formula A, $x :^{\bullet} A \in \Gamma$ implies $x \vDash^{\bullet} A$ while $x :^{\bullet} A \in \Delta$ implies $x \nvDash^{\bullet} A$, yielding a countermodel to $\Gamma \Rightarrow \Delta$.

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- Define a frame $\mathscr{F}$ on the labels occurring in Γ within the branch.
- Show that, for every formula A, $x :^{\bullet} A \in \Gamma$ implies $x \vDash^{\bullet} A$ while $x :^{\bullet} A \in \Delta$ implies $x \nvDash^{\bullet} A$, yielding a countermodel to $\Gamma \Rightarrow \Delta$.

Corollary. (Completeness)

If $\Gamma \Rightarrow \Delta$ is C-valid, then $\vdash_{\mathsf{GC}^{+}_{-}} \Gamma \Rightarrow \Delta$.

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Sources of non termination

- Relational rules:

$$\frac{x \leqslant x, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \text{REF} \qquad \frac{x \leqslant z, x \leqslant y, y \leqslant z, \Gamma \Rightarrow \Delta}{x \leqslant y, y \leqslant z, \Gamma \Rightarrow \Delta}\ \text{TRS}$$

- Contraction-absorbing rules:

$$\frac{x \leqslant y, x :^{+} A \rightarrow B, \Gamma \Rightarrow \Delta, y :^{+} A \qquad y :^{+} B, x \leqslant y, x :^{+} A \rightarrow B, \Gamma \Rightarrow \Delta}{x \leqslant y, x :^{+} A \rightarrow B, \Gamma \Rightarrow \Delta}\ L_{\rightarrow}^{+}$$

$$\frac{x \leqslant y, x :^{-} A \rightarrow B, \Gamma \Rightarrow \Delta, y :^{+} A \qquad y :^{-} B, x \leqslant y, x :^{-} A \rightarrow B, \Gamma \Rightarrow \Delta}{x \leqslant y, x :^{-} A \rightarrow B, \Gamma \Rightarrow \Delta}\ L_{\rightarrow}^{-}$$

- Rules with eigenvariable condition:

$$(y \text{ fresh})\ \frac{x \leqslant y, y :^{+} A, \Gamma \Rightarrow \Delta, y :^{+} B}{\Gamma \Rightarrow \Delta, x :^{+} A \rightarrow B}\ R_{\rightarrow}^{+} \qquad (y \text{ fresh})\ \frac{x \leqslant y, y :^{+} A, \Gamma \Rightarrow \Delta, y :^{-} B}{\Gamma \Rightarrow \Delta, x :^{+} A \rightarrow B}\ R_{\rightarrow}^{-}$$

Example

- Attempt at a derivation of $\Rightarrow x :^{+} ((A \to B) \to C) \to C$

$$(\pi)$$

$$\vdots$$

$$\dfrac{y \leqslant u, z :^{+} A, y \leqslant y, x \leqslant y, z \leqslant u, u :^{+} A, y \leqslant z, \Theta \Rightarrow y :^{+} C, z :^{+} B, u :^{+} B, u :^{+} A \to B \qquad \Lambda_3}{y \leqslant u, z :^{+} A, y \leqslant y, x \leqslant y, z \leqslant u, u :^{+} A, y \leqslant z, \Theta \Rightarrow y :^{+} C, z :^{+} B, u :^{+} B}\; L^{+}_{\to}$$

$$\dfrac{y \leqslant u, z :^{+} A, y \leqslant y, x \leqslant y, z \leqslant u, u :^{+} A, y \leqslant z, \Theta \Rightarrow y :^{+} C, z :^{+} B, u :^{+} B}{z \leqslant u, u :^{+} A, y \leqslant z, z :^{+} A, y \leqslant y, x \leqslant y, \Theta \Rightarrow y :^{+} C, z :^{+} B, u :^{+} B}\; \mathrm{TRS}$$

$$\dfrac{z \leqslant u, u :^{+} A, y \leqslant z, z :^{+} A, y \leqslant y, x \leqslant y, \Theta \Rightarrow y :^{+} C, z :^{+} B, u :^{+} B}{y \leqslant z, z :^{+} A, y \leqslant y, x \leqslant y, \Theta \Rightarrow y :^{+} C, z :^{+} B, z :^{+} A \to B}\; R^{+}_{\to}$$

$$\dfrac{y \leqslant z, z :^{+} A, y \leqslant y, x \leqslant y, \Theta \Rightarrow y :^{+} C, z :^{+} B, z :^{+} A \to B \qquad \Lambda_2}{y \leqslant z, z :^{+} A, y \leqslant y, x \leqslant y, \Theta \Rightarrow y :^{+} C, z :^{+} B}\; L^{+}_{\to}$$

$$\dfrac{y \leqslant z, z :^{+} A, y \leqslant y, x \leqslant y, \Theta \Rightarrow y :^{+} C, z :^{+} B}{y \leqslant y, x \leqslant y, \Theta \Rightarrow y :^{+} C, y :^{+} A \to B}\; R^{+}_{\to}$$

$$\dfrac{y \leqslant y, x \leqslant y, \Theta \Rightarrow y :^{+} C, y :^{+} A \to B \qquad \Lambda_1}{y \leqslant y, x \leqslant y, y :^{+} (A \to B) \to C \Rightarrow y :^{+} C}\; L^{+}_{\to}$$

$$\dfrac{y \leqslant y, x \leqslant y, y :^{+} (A \to B) \to C \Rightarrow y :^{+} C}{x \leqslant y, y :^{+} (A \to B) \to C \Rightarrow y :^{+} C}\; \mathrm{REF}$$

$$\dfrac{x \leqslant y, y :^{+} (A \to B) \to C \Rightarrow y :^{+} C}{\Rightarrow x :^{+} ((A \to B) \to C) \to C}\; R^{+}_{\to}$$

$(\Theta = y :^{+} (A \to B) \to C.)$

- The pattern can be repeated indefinitely (cyclic search):
  1. Repeated applications of $L^{+}_{\to}$ facilitate the use of $R^{+}_{\to}$;
  2. This allows for the application of TRS;
  3. Apply another time $L^{+}_{\to}$ to the copy of the principal formula Θ.

Strategy

- To establish finiteness of the proof-search space, we restrict attention to minimal derivations, i.e. derivations admitting no shortenings.

1. Proof-theoretic result based on the properties of GC^{+}_{-}, i.e., hp-admissibility of label substitution and contraction.
2. Notion of saturation of a tree defined by the exhaustive application of all available rules, except those that would introduce redundancies – such as loops or duplications of existing formulas – modulo label substitution.
3. Construction of a countermodel.
4. Computational complexity of the decision procedure.

Source of non-termination 1 (relational rules)

- Although the rule TRS can in principle be applied infinitely often to the same principal formulas, this does not occur in minimal derivations.
- Repeated applications of relational rules to the same principal formulas would violate minimality (modulo substitution and contraction).

Lemma.

All variables in atoms $x \leqslant x$ removed by applications of REF in a minimal derivation of a sequent $\Gamma \Rightarrow \Delta$ in GC^{+}_{-} are variables in Γ, Δ.

Proposition. (Subterm Property)

All variables in a minimal derivation of a sequent $\Gamma \Rightarrow \Delta$ in GC^{+}_{-} are either eigenvariables or variables in Γ, Δ.

Source of non-termination 2 (contraction-absorbing rules)

$$
\begin{array}{c}
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\begin{array}{c}
\dfrac{\begin{array}{c}\vdots \\ x \leqslant y, x :^{\bullet} A \to B, \Gamma'' \Rightarrow \Delta''\end{array} \quad \begin{array}{c}\vdots \\ x \leqslant y, x :^{\bullet} A \to B, \Gamma'' \Rightarrow \Delta''\end{array}}{x \leqslant y, x :^{\bullet} A \to B, \Gamma' \Rightarrow \Delta'}\, L^{\bullet}_{\to} \\
\vdots \\
x \leqslant y, x :^{\bullet} A \to B, \Gamma' \Rightarrow \Delta'
\end{array}
\quad
\begin{array}{c}
\dfrac{\begin{array}{c}\vdots \\ x \leqslant y, x :^{\bullet} A \to B, \Gamma'' \Rightarrow \Delta''\end{array} \quad \begin{array}{c}\vdots \\ x \leqslant y, x :^{\bullet} A \to B, \Gamma'' \Rightarrow \Delta''\end{array}}{x \leqslant y, x :^{\bullet} A \to B, \Gamma' \Rightarrow \Delta'}\, L^{\bullet}_{\to} \\
\vdots \\
x \leqslant y, x :^{\bullet} A \to B, \Gamma' \Rightarrow \Delta'
\end{array}
}{x \leqslant y, x :^{\bullet} A \to B, \Gamma \Rightarrow \Delta}\, L^{\bullet}_{\to} \\
\vdots
\end{array}
$$

Source of non-termination 2 (contraction-absorbing rules)

$$
\dfrac{\dfrac{\dfrac{\vdots}{x \leqslant y, x :^{\bullet} A \to B, \Gamma'' \Rightarrow \Delta''} \quad \dfrac{\vdots}{x \leqslant y, x :^{\bullet} A \to B, \Gamma'' \Rightarrow \Delta''}}{\dfrac{x \leqslant y, x :^{\bullet} A \to B, \Gamma' \Rightarrow \Delta'}{\dfrac{\vdots}{x \leqslant y, x :^{\bullet} A \to B, \Gamma' \Rightarrow \Delta'}}} L^{\bullet}_{\to} \quad \dfrac{\dfrac{\vdots}{x \leqslant y, x :^{\bullet} A \to B, \Gamma'' \Rightarrow \Delta''} \quad \dfrac{\vdots}{x \leqslant y, x :^{\bullet} A \to B, \Gamma'' \Rightarrow \Delta''}}{\dfrac{x \leqslant y, x :^{\bullet} A \to B, \Gamma' \Rightarrow \Delta'}{\dfrac{\vdots}{x \leqslant y, x :^{\bullet} A \to B, \Gamma' \Rightarrow \Delta'}}} L^{\bullet}_{\to}}{\dfrac{x \leqslant y, x :^{\bullet} A \to B, \Gamma \Rightarrow \Delta}{\vdots}} L^{\bullet}_{\to}
$$

Lemma. (Permutation)

The rules $L^{+}_{\to}, L^{-}_{\to}$ permute down with respect to $L^{+}_{\sim}$, $L^{-}_{\sim}$, $R^{+}_{\sim}$, $R^{-}_{\sim}$, $L^{+}_{\vee}$, $L^{-}_{\vee}$, $R^{+}_{\vee}$, $R^{-}_{\vee}$, $L^{+}_{\wedge}$, $L^{-}_{\wedge}$, $R^{+}_{\wedge}$, $R^{-}_{\wedge}$. Furthermore, they permute down with instances of $R^{+}_{\to}$, $R^{-}_{\to}$, as well as with relational rules, provided the principal atom of $L^{+}_{\to}$, $L^{-}_{\to}$ is not active in $R^{+}_{\to}$, $R^{-}_{\to}$ or the relational rules.

Proposition. (Bound on $L^{+}_{\to}, L^{-}_{\to}$)

In a minimal derivation in GC^{+}_{-}, rules $L^{+}_{\to}, L^{-}_{\to}$ cannot be applied more than once on the same pair of principal formulas on any branch.

Source of non-termination 3 (rules with eigenvariable)

- We follow the proof-search procedure from the completeness proof, applying the rules root-first until a saturation condition is reached.
- A branch is saturated when its leaf is not an initial sequent and it is closed under all rules of the calculus.
- The only exceptions are applications of the right verification and falsification rules for $\rightarrow$ that would generate loops (modulo fresh labels).
- From a saturated branch, we construct a finite countermodel.

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- The only exceptions are applications of the right verification and falsification rules for $\rightarrow$ that would generate loops (modulo fresh labels).
- From a saturated branch, we construct a finite countermodel.
- The underlying partial order is obtained as the reflexive and transitive closure of $\leqslant$, extended with a relation accounting for the looping behavior induced by certain rule applications.

Definition.

For a sequent $\Gamma \Rightarrow \Delta$ in a proof search tree and a label x, we indicate with $\mathtt{F}_{\Gamma\Rightarrow\Delta}(x)$ the ordered pair of sets $\langle \mathtt{F}^1_{\Gamma\Rightarrow\Delta}(x), \mathtt{F}^2_{\Gamma\Rightarrow\Delta}(x)\rangle$, where:

$$\mathtt{F}^1_{\Gamma\Rightarrow\Delta}(x) \equiv \{A \mid x :^{\bullet} A \in \downarrow\Gamma\} \cup \{p \mid y :^{\bullet} p, y \leqslant x \in \Gamma\} \cup \{A \rightarrow B \mid y :^{\bullet} A \rightarrow B, y \leqslant x \in \Gamma\},$$

$$\mathtt{F}^2_{\Gamma\Rightarrow\Delta}(x) \equiv \{A \mid x :^{\bullet} A \in \downarrow\Delta\}.$$

Finally, $x \preccurlyeq_{\Gamma\Rightarrow\Delta} y$ iff $\mathtt{F}^i_{\Gamma\Rightarrow\Delta}(x) \subseteq \mathtt{F}^i_{\Gamma\Rightarrow\Delta}(y)$, for $i = 1, 2$.

Definition. (Saturation conditions)

A branch in a proof search up to a GC_{-}^{+} sequent $\Gamma \Rightarrow \Delta$ is saturated if the following conditions hold:

- If x is a label in Γ, Δ, then $x \leqslant x$ is in Γ.
- If $x \leqslant y$ and $y \leqslant z$ are in Γ, then $x \leqslant z$ is in Γ.
- If $x :^{-} p$ is in Γ, there is no y such that $x \leqslant y$ is in Γ and $y :^{-} p$ is in Δ.
- If $x :^{-} {\sim}A$ is in $\downarrow\Gamma$, then $x :^{+} A$ is in $\downarrow\Gamma$.
- If $x :^{-} {\sim}A$ is in $\downarrow\Delta$, then $x :^{+} A$ is in $\downarrow\Delta$.
- If $x :^{-} A \wedge B$ is in $\downarrow\Gamma$, then either $x :^{-} A$ is in $\downarrow\Gamma$ or $x :^{-} B$ is in $\downarrow\Gamma$.
- If $x :^{-} A \wedge B$ is in $\downarrow\Delta$, then $x :^{-} A$ is in $\downarrow\Delta$ and $x :^{-} B$ is in $\downarrow\Delta$.
- If $x :^{-} A \vee B$ is in $\downarrow\Gamma$, then $x :^{-} A$ is in $\downarrow\Gamma$ and $x :^{-} B$ is in $\downarrow\Gamma$.
- If $x :^{-} A \vee B$ is in $\downarrow\Delta$, then either $x :^{-} A$ is in $\downarrow\Delta$ or $x :^{-} B$ is in $\downarrow\Delta$.
- If $x :^{-} A \rightarrow B$ and $x \leqslant y$ are in Γ, then either $y :^{+} A$ is in $\downarrow\Delta$ or $y :^{-} B$ is in $\downarrow\Gamma$.
- If $x :^{-} A \rightarrow B$ is in $\downarrow\Delta$, then either
  - for some y, there is $x \leqslant y$ in Γ, $y :^{+} A$ is in $\downarrow\Gamma$, and $y :^{-} B$ is in $\downarrow\Delta$, or
  - there exists $y \neq x$ such that $y \leqslant x$ is in Γ and $x \preccurlyeq_{\Gamma \Rightarrow \Delta} y$.

Countermodel construction

- Build the countermodel by starting from a saturated branch and use the two finite sets $\downarrow\Gamma, \downarrow\Delta$.

Given a sequent $\Gamma \Rightarrow \Delta$, we define the countermodel $\mathscr{M} = \langle W, \leqslant, \mathrm{v}^+, \mathrm{v}^- \rangle$ as follows:

- The set W consists precisely of all labels occurring in Γ.
- The relation $\leqslant$ is the reflexive and transitive closure of the union of the order relations in Γ together with the relation $\preccurlyeq_{\Gamma\Rightarrow\Delta}$.
- The forcing relation is defined on atomic formulas by $x \vDash^{\bullet} p$ if there are both $y \leqslant x$ and $y :^{\bullet} p$ in Γ. This relation is then extended to arbitrary formulas according to the standard clauses of the relational semantics for C.

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Proposition.

Let $\bullet \in \{+, -\}$. The following results hold:

1. If $\mathscr{M}, x \vDash^{\bullet} p$ and $x \leqslant y$, then $\mathscr{M}, y \vDash^{\bullet} p$.

2a. If $A \in \mathtt{F}^1(x)$, then $\mathscr{M}, x \vDash^{\bullet} A$.

2b. If $A \in \mathtt{F}^2(x)$, then $\mathscr{M}, x \nvDash^{\bullet} A$.

Example (Blocking the derivation)

- Consider the indefinitely proceeding proof search (π), and take its topsequent:

(Σ) $y \leqslant u,\ z :^{+} A,\ y \leqslant y,\ x \leqslant y,\ z \leqslant u,\ u :^{+} A,\ y \leqslant z,\ y :^{+} (A \to B) \to C \Rightarrow y :^{+} C,\ z :^{+} B,$
$u :^{+} B,\ u :^{+} A \to B.$

- Let $\mathtt{F}_{\Sigma}(u)$ and $\mathtt{F}_{\Sigma}(z)$ be the ordered pairs of sets $\langle \mathtt{F}^{1}_{\Sigma}(u), \mathtt{F}^{2}_{\Sigma}(u)\rangle$ and $\langle \mathtt{F}^{1}_{\Sigma}(z), \mathtt{F}^{2}_{\Sigma}(z)\rangle$, respectively. These are defined as follows: $\mathtt{F}^{1}_{\Sigma}(u) = \{A, (A \to B) \to C\}$, $\mathtt{F}^{2}_{\Sigma}(u) = \{B\}$, $\mathtt{F}^{1}_{\Sigma}(z) = \{A, (A \to B) \to C\}$, and $\mathtt{F}^{2}_{\Sigma}(z) = \{B\}$. Hence, we have that $\mathtt{F}_{\Sigma}(u) = \langle \{A, (A \to B) \to C\}, \{B\}\rangle = \mathtt{F}_{\Sigma}(z)$.

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- Let $\mathtt{F}_{\Sigma}(u)$ and $\mathtt{F}_{\Sigma}(z)$ be the ordered pairs of sets $\langle \mathtt{F}^{1}_{\Sigma}(u), \mathtt{F}^{2}_{\Sigma}(u) \rangle$ and $\langle \mathtt{F}^{1}_{\Sigma}(z), \mathtt{F}^{2}_{\Sigma}(z) \rangle$, respectively. These are defined as follows: $\mathtt{F}^{1}_{\Sigma}(u) = \{A, (A \to B) \to C\}$, $\mathtt{F}^{2}_{\Sigma}(u) = \{B\}$, $\mathtt{F}^{1}_{\Sigma}(z) = \{A, (A \to B) \to C\}$, and $\mathtt{F}^{2}_{\Sigma}(z) = \{B\}$. Hence, we have that $\mathtt{F}_{\Sigma}(u) = \langle \{A, (A \to B) \to C\}, \{B\} \rangle = \mathtt{F}_{\Sigma}(z)$.
- We first apply $\textsc{sub}(z/u)$. We then eliminate any resulting duplications of formulas and relational atoms via successive applications of $\textsc{c}$.

  (Σ') $\vdash^{n} y \leqslant y, x \leqslant y, z \leqslant z, z :^{+} A, y \leqslant z, y :^{+} (A \to B) \to C \Rightarrow y :^{+} C, z :^{+} B, z :^{+} A \to B.$

  Recall that applications of both $\textsc{sub}$ and $\textsc{c}$ are height-preserving. Furthermore, we apply $\textsc{ref}$ on $z \leqslant z$ within (Σ') and obtain the following sequent of height $n+1$:

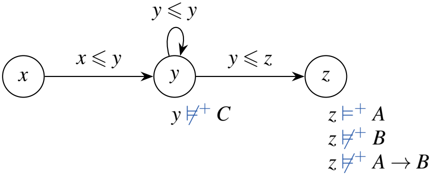
  (Σ'') $\vdash^{n+1} y \leqslant y, x \leqslant y, z :^{+} A, y \leqslant z, y :^{+} (A \to B) \to C \Rightarrow y :^{+} C, z :^{+} B, z :^{+} A \to B.$
- Start from (Σ'') to construct a shorter derivation of $\Rightarrow x :^{+} ((A \to B) \to C) \to C$. Violation of minimality!

Example (Countermodel extraction)

- Construct a countermodel to $\Rightarrow x :^{+} ((A \to B) \to C) \to C$.

(Σ'') $y \leqslant y,\ x \leqslant y,\ z :^{+} A,\ y \leqslant z,\ y :^{+} (A \to B) \to C \Rightarrow y :^{+} C,\ z :^{+} B,\ z :^{+} A \to B.$

- Let $W = \{x, y, z\}$, and let $\leqslant$ be the preorder generated by the relations $x \leqslant y$ and $y \leqslant z$. Let the valuation be defined so that $z \vDash^{+} A$ but $z \nvDash^{+} B$, and $z \nvDash^{+} A \to B$. Assume also that $y \nvDash^{+} C$. We examine the formula $((A \to B) \to C) \to C$ at world x. Since $x \leqslant y$, and forcing in our models is upward monotonic, it suffices to check whether $y \vDash^{+} (A \to B) \to C$.



Computational Complexity

- We establish an explicit operational upper bound on the complexity of proof search in GC^{+}_{-}.
- The decision problem for C is solvable in ExpTime and ExpSpace.

Computational Complexity

- We establish an explicit operational upper bound on the complexity of proof search in GC^{+}_{-}.
- The decision problem for C is solvable in $\textsc{ExpTime}$ and $\textsc{ExpSpace}$.
- Let $\Gamma \Rightarrow \Delta$ be a fixed end-sequent and let $m = |\mathsf{sbf}(\Gamma \Rightarrow \Delta)|$ be the cardinality of its (weak) subformula closure.

Lemma. (Label bound)

In every minimal root-first proof search of a sequent $\Gamma \Rightarrow \Delta$ in GC^{+}_{-}, at most 4^m distinct labels can be introduced on any branch.

Theorem. (Complexity)

1. Root-first proof search in GC^{+}_{-} terminates in time $2^{O(m)}$.
2. Root-first proof search in GC^{+}_{-} runs in space $2^{O(m)}$.

Contents

1. Connexivity and Bilateralism
2. Proof Theory & Completeness
3. Termination, Countermodel & Complexity
4. Conclusion

Final remarks

- Connexive and contradictory character of C:

Provable formulas.

- $\vdash_{\mathsf{GC}^{+}_{-}} \Rightarrow x :^{+} \sim(A \leftrightarrow \sim A)$,
- $\vdash_{\mathsf{GC}^{+}_{-}} \Rightarrow x :^{+} \sim(A \to B) \leftrightarrow (A \to \sim B)$,
- $\vdash_{\mathsf{GC}^{+}_{-}} \Rightarrow x :^{+} (A \wedge \sim A) \to A$ and $\vdash_{\mathsf{GC}^{+}_{-}} \Rightarrow x :^{+} \sim((A \wedge \sim A) \to A)$,
- $\vdash_{\mathsf{GC}^{+}_{-}} \Rightarrow x :^{+} \sim A \to \sim(A \to \sim A)$ and $\vdash_{\mathsf{GC}^{+}_{-}} \Rightarrow x :^{+} \sim(\sim A \to \sim(A \to \sim A))$,
- $\nvdash_{\mathsf{GC}^{+}_{-}} \Rightarrow x :^{+} (A \to B) \to (B \to A)$.

- The same results can be adapted to N4 and C3.

Thank you!

Děkuji!

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