Deductive Interpolation in Hájek's Basic Fuzzy Logic

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Interpolation properties

Interpolation is one of the fundamental metalogical properties, and comes in many different forms.

Craig interpolation property (CIP):

$$\alpha \to \beta \Rightarrow \exists \delta(\mathsf{var}(\delta) \subseteq \mathsf{var}(\alpha) \cap \mathsf{var}(\beta), \alpha \to \delta, \delta \to \beta)$$

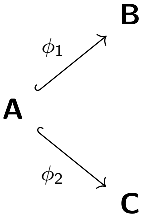
Deductive interpolation property (DIP):

$$\alpha \vdash \beta \Rightarrow \exists \delta(\mathsf{var}(\delta) \subseteq \mathsf{var}(\alpha) \cap \mathsf{var}(\beta), \alpha \vdash \delta, \delta \vdash \beta)$$

Deductive interpolation (or, more correctly, the Robinson consistency theorem) is associated very strongly to amalgamation on the level of algebraic models: Under good hypotheses, a logic has DIP iff its class of algebraic models has the amalgamation property.

The amalgamation property

A class K of algebras has the amalgamation property if every span $\langle \phi_1 \colon \mathbf{A} \to \mathbf{B}, \phi_2 \colon \mathbf{A} \to \mathbf{C} \rangle$ of algebras in K can be completed in K:



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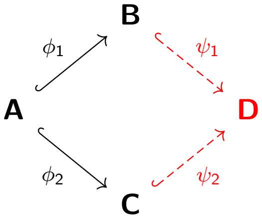

Interpolation and amalgamation around BL

Quite a history of trying to pin extensions of Hájek's basic fuzzy logic BL that have interpolation.

[Montagna 2006]: Variety of all BL-algebras + many of the most natural subvarieties have AP, but there are uncountably many that do not. Very few extensions of basic logic with CIP.

Montagna's problem: How many varieties of BL-algebras have AP? Countably many or uncountably many?

[Cortonesi-Marchioni-Montagna 2011]: Applied tools from first-order model theory.

[Aguzzoli-Bianchi 2021]: Partial classification for finitely generated varieties.

[Fussner-Metcalfe 2022]: New general results for studying AP.

[Aguzzoli-Bianchi 2023]: Sharpened classification, but still not complete.

The poset $\Omega(V)$

If V is any variety, the subvarieties of V form a lattice $\Lambda(V)$ and we can consider the subposet of $\Omega(V)$ of $\Lambda(V)$ consisting of only those subvarieties with the AP.

Not much is known about $\Omega(V)$ for arbitrary V.

But a lot of the literature on interpolation/amalgamation describes various features of $\Omega(V)$ for particular V.

E.g. Maksimova's Theorem says that $|\Omega(HA)| = 8$.

MV-algebras and Wajsberg hoops

A Wajsberg hoop is an integral commutative semilinear residuated lattice that satisfies $(x \to y) \to y = x \vee y$, i.e. a bottom-free subreduct of an MV-algebra. Denote by:

- $Ł_n$ the n-element MV-algebra chain.
- $\mathbf{Z}$ the negative cone of the integers (a cancellative hoop)
- $\mathbf{W}_m$ the 0-free reduct of the MV-algebra $Ł_m$.
- $\mathbf{W}_{m,\omega}$ the 0-free reduct of the MV-algebra $\Gamma(\mathbb{Z} \times \mathbb{Z}, \langle m, 0 \rangle)$, where $\mathbb{Z} \times \mathbb{Z}$ is ordered lexicographically as an ℓ-group and Γ is the Mundici functor.

[Di Nola-Lettieri 2000]: A variety of MV-algebras has the AP if and only if it is generated by a single totally ordered MV-algebra.

[Metcalfe-Montagna-Tsinakis 2014]: A variety of Wajsberg hoops has the AP if and only if it has the form $\mathbb{V}(\mathbf{W}_m)$, $\mathbb{V}(\mathbf{W}_{m,\omega})$, $\mathbb{V}(\mathbf{Z})$, $\mathbb{V}(\mathbf{W}_m, \mathbf{Z})$, or WH.

BL-algebras and basic hoops

A BL-algebra is a pointed commutative integral residuated lattice (with additional constant 0 for the bottom element) satisfying $(x \to y) \vee (y \to x) \approx 1$ and $x \cdot (x \to y) \approx x \wedge y$. A basic hoop is a 0-free subreduct of a BL-algebra. These algebras are all semilinear.

[Aglianò-Montagna 2003]: Every totally ordered basic hoop $\mathbf{A}$ is an ordinal sum $\mathbf{A} \cong \bigoplus_{i \in I} \mathbf{A}_i$ of totally ordered Wajsberg hoops. The same representation holds for BL-algebras by adding a totally ordered MV-algebra.

If I is finite, then $\mathbf{A}$ has finite index. Every variety V of basic hoops is generated by its finite index members V_{fc}. It is enough to consider these to study the AP.

The strategy

- Start with basic hoops, and then use the result for BL-algebras
- For each variety V of basic hoops, reduce whether $V \in \Omega(BH)$ to the totally ordered members of V
- Use the ordinal sum decomposition of totally ordered basic hoops into their Wajsberg hoop components [Aglianó-Montagna 2003] to further reduce the question to whether the components have AP
- Varieties of Wajsberg hoops with the AP are already classified [Metcalfe-Montagna-Tsinakis 2014]
- This is a start, but not enough: Need to get into how the Wajsberg components are arranged in these totally ordered basic hoops
- Combine with the characterization of varieties of MV-algebras with the AP [Di Nola-Lettieri 2000] and the ordinal sum decomposition for BL-algebras to get the result for BL

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First point: the essential AP

An extension $\mathbf{A} \leq \mathbf{B}$ is essential if for every congruence $\theta \neq \Delta_{\mathbf{B}}$ of $\mathbf{B}$, $\theta \cap A^2 \neq \Delta_{\mathbf{A}}$.

An embedding $\varphi\colon \mathbf{A} \to \mathbf{B}$ is essential if $\varphi[\mathbf{A}] \leq \mathbf{B}$ is. Equivalently, this holds if for each $\psi\colon \mathbf{B} \to \mathbf{C}$, if $\psi \circ \phi$ is an embedding implies that ψ is an embedding.

A span $\langle \phi_B\colon \mathbf{A} \to \mathbf{B}, \phi_C\colon \mathbf{A} \to \mathbf{C} \rangle$ is essential if ϕ_C is essential, and a class K has the essential AP if every essential span in K has an amalgam in K.

Theorem (F.-Santschi 2024):

Let V be any variety of (pointed) semilinear residuated lattices with the CEP. Then V has the AP if and only if every essential span of finitely generated chains in V has a totally ordered amalgam in V.

Useful nomenclature

We introduce some naming conventions for varieties:

- $\mathbf{A} \oplus \mathbf{B}$ is written $\mathbf{AB}$
- The class generated by the componentwise $\mathrm{HSP_u}$ closure of an ordinal sum by enclosing the corresponding ordinal sum in bracket [,], so that, for example, $[\mathbf{AB}]$ denotes the class of all ordinal sums $\mathbf{A}' \oplus \mathbf{B}'$ where $\mathbf{A}' \in \mathrm{HSP_u}(\mathbf{A})$ and $\mathbf{B}' \in \mathrm{HSP_u}(\mathbf{B})$; and $[\mathbf{A}] = \mathrm{HSP_u}(\mathbf{A})$.
- We use * to denote the repetition of one or more instances of a summand in a given ordinal sum. For example, $[\mathbf{AB}^*]$ abbreviates the class consisting of all ordinal sums of the form $\mathbf{A} \oplus \mathbf{B}_1 \oplus \cdots \oplus \mathbf{B}_n$, where n is a positive integer and $\mathbf{B}_1, \ldots, \mathbf{B}_n \in \mathrm{HSP_u}(\mathbf{B})$.
- Kleene star * has priority over $\oplus$, so that $[\mathbf{ABC}^*]$ abbreviates $[(\mathbf{A} \oplus \mathbf{B}) \oplus \mathbf{C}^*]$.

A decomposition into intervals

For each variety V of basic hoops, denote by *Wajs*(V) the class of Wajsberg chains in V. If $\mathsf{V} \in \Omega(\mathsf{BH})$, then $\mathbb{V}(\mathit{Wajs}(\mathsf{V})) \in \Omega(\mathsf{WH})$.

Theorem (F.-Santschi 2025):

The poset $\Omega(\mathsf{BH})$ can be partitioned into countably infinitely many closed intervals: for any variety V of basic hoops with the amalgamation property one of the following holds.

1. V is trivial.
2. $\mathrm{Wajs}(\mathsf{V}) = [\mathbf{A}]$ for $\mathbf{A} \in \{\mathbf{W}_n \mid n \geq 1\} \cup \{\mathbf{Z}, [0,1]_{\mathsf{WH}}\}$, and $[\mathbf{A}] \subseteq \mathsf{V}_{\mathrm{fc}} \subseteq [\mathbf{A}^*]$.
3. $\mathrm{Wajs}(\mathsf{V}) = [\mathbf{W}_{n,\omega}]$ for some $n \geq 1$, and $[\mathbf{W}_{n,\omega}] \subseteq \mathsf{V}_{\mathrm{fc}} \subseteq [\mathbf{W}^*_{n,\omega}]$.
4. $\mathrm{Wajs}(\mathsf{V}) = [\mathbf{W}_n] \cup [\mathbf{Z}]$ for some $n \geq 1$, and $[\mathbf{W}_n] \cup [\mathbf{Z}] \subseteq \mathsf{V}_{\mathrm{fc}} \subseteq [(\mathbf{W}_n\mathbf{Z})^*]$.

The second point: closure properties

Lemma (F.-Santschi 2025):

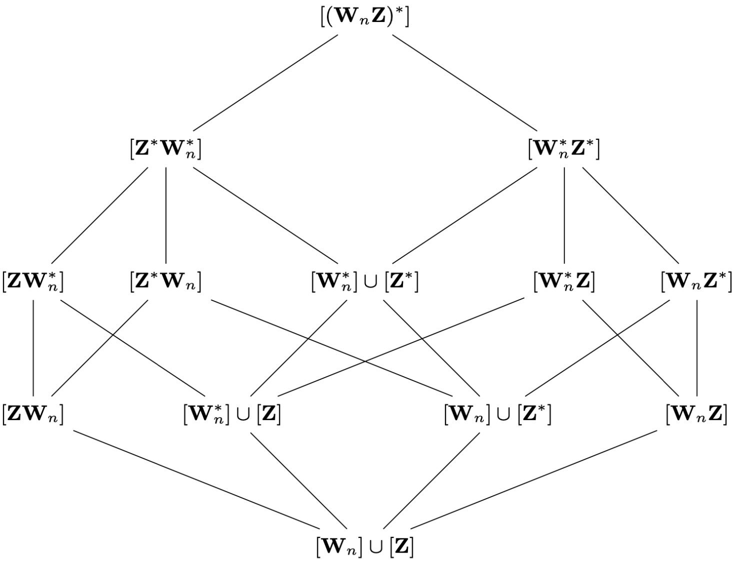
Let $\mathsf{V} \in \Omega(\mathsf{BH})$. If $\bigoplus_{i=1}^{l} \mathbf{A}_i \in \mathsf{V}_{\mathrm{fc}}$ with $\mathbf{A}_i \in \mathrm{Wajs}(\mathsf{V})$, then $\bigoplus_{i=1}^{l} \mathrm{H}(\mathbf{A}_i) \subseteq \mathsf{V}_{\mathrm{fc}}$.

Lemma (F.-Santschi 2025):

Let $\mathsf{V} \in \Omega(\mathsf{BH})$ and $\mathbf{A}, \mathbf{B}, \mathbf{C}$ Wajsberg chains.

1. If $\mathbf{A} \oplus \mathbf{B} \oplus \mathbf{B} \oplus \mathbf{C} \in \mathsf{V}_{\mathrm{fc}}$, then $[\mathbf{A}\mathbf{B}^*\mathbf{C}] \subseteq \mathsf{V}_{\mathrm{fc}}$.
2. If $\mathbf{W}_n \oplus \mathbf{W}_{n,\omega} \in \mathsf{V}_{\mathrm{fc}}$, then $[\mathbf{W}_n^*\mathbf{W}_{n,\omega}] \subseteq \mathsf{V}_{\mathrm{fc}}$.
3. If $\mathbf{A} \oplus \mathbf{A} \oplus \mathbf{B} \oplus \mathbf{B} \in \mathsf{V}_{\mathrm{fc}}$ or $\mathbf{A} \oplus \mathbf{A}, \mathbf{A} \oplus \mathbf{B}, \mathbf{B} \oplus \mathbf{B} \in \mathsf{V}_{\mathrm{fc}}$, then $[\mathbf{A}^*\mathbf{B}^*] \subseteq \mathsf{V}_{\mathrm{fc}}$.
4. If $\mathbf{A} \oplus \mathbf{B}, \mathbf{B} \oplus \mathbf{A} \in \mathsf{V}_{\mathrm{fc}}$, then $[(\mathbf{A}\mathbf{B})^*] \subseteq \mathsf{V}_{\mathrm{fc}}$.

Charting Ω(BH)



Taking stock

Theorem (F.-Santschi 2025):

The poset $\Omega(\mathrm{BH})$ can be partitioned into countably many finite intervals, and hence there are only countably many axiomatic extensions of falsum-free basic logic with the DIP.

Theorem (F.-Santschi 2025):

The poset $\Omega(\mathrm{BL})$ can be partitioned into countably many finite intervals, and hence there are only countably many axiomatic extensions of Hájek's basic logic with the DIP.

Looking forward

We have seen that the essential AP combines with the structure theory of basic hoops/BL-algebras to yield a Maksimova-type classification of logics with DIP.

It should be stressed that the structural description is not enough.

Tools for studying AP that are tailored to the situation are also essential, as well as an understanding of concrete obstructions.

The same classification strategy works in many other contexts: Certain subvarieties of MTL, varieties whose SI members that can be decomposed as a disjoint union over the fibers of a (co-)nucleus, etc.

For more information, see:

W. Fussner and S. Santschi, Interpolation in Hájek's Basic Logic, Annals of Pure and Applied Logic 176, paper 103615 (2025).

Thank you!