Interpolation properties among arbitrary extensions of RM

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Interpolation in non-classical logics, relevance logics particularly

- Maksimova's 1977 characterization: Exactly seven superintuitionistic logics have CIP
- some classes of substructural logics, particularly semilinear (Fussner Santschi, 2024, 2025)
- Slogan for the common knowledge "Interpolation fails in relevant logics" (Urquhart 1993, 1984);
- some positive results: FDE has PCIP (Anderson Belnap 1975);
- substructurals with the constants: RM^t has CIP (Meyer 1980).
- R-mingle with the Ackermann constant (RM^t) (Marchioni-Metcalfe 2012) exactly eight non-trivial extensions.

Limitations and challenges

- known results restricted to axiomatic extensions (varieties);
- no progress in terms of arbitrary extensions (quasivarieties);
- Goal: get a deeper understanding of interpolation-like properties in non-classical logic;
- R–mingle as a good case study (historically well-known system, nice semantics - Sugiharas).

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RM fails the imperfect CIP

DIP, MIP and Robinson property

If $\Gamma \vdash \beta$ and $\text{var}(\Gamma) \cap \text{var}(\beta) \neq \emptyset$, then there is a set of formulas Δ such that $\text{var}(\Delta) \subseteq \text{var}(\Gamma) \cap \text{var}(\beta)$ and both $\Gamma \vdash \Delta$ and $\Delta \vdash \beta$. (DIP)

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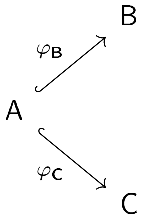
Whenever $X, Y \subseteq \mathrm{var}$ such that $X \cap Y \neq \emptyset$, T is a theory of L over X, and S is a theory of L over Y such that $T \cap \mathrm{Fm}_{\mathcal{L}}(X \cap Y) = S \cap \mathrm{Fm}_{\mathcal{L}}(X \cap Y)$, there exists a theory R of L over $X \cup Y$ such that $T = R \cap \mathrm{Fm}_{\mathcal{L}}(X)$ and $S = R \cap \mathrm{Fm}_{\mathcal{L}}(Y)$. (RP)

Amalgamation Property

class K has AP iff every span $\langle \varphi_B : A \hookrightarrow B, \varphi_C : A \hookrightarrow C \rangle$ in K can be completed in K (has an amalgam in K) i.e. there exists $D \in K$ and embeddings $\langle \psi_B : B \hookrightarrow D, \psi_C : C \hookrightarrow D \rangle$ such that $\psi_B \varphi_B = \psi_C \varphi_C$.



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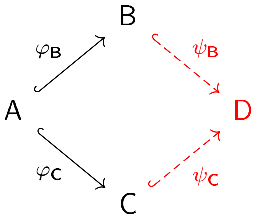
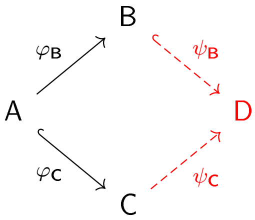


Transferable injections TIP – just diagramatically



A known algebraic fact:
TIP $\Leftrightarrow$ AP + RCEP

AAL characterization Czelakowski-Pigozzi 1999

AP $\Leftrightarrow$ RP
TIP $\Leftrightarrow$ MIP
Thus, as a corollary we have AP + RCEP $\Leftrightarrow$ MIP

the quasivariety challenge

Results on amalgamation predominantly restricted to varieties.
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(1) We usually have a bad handle on R(F)SI's in quasivarieties (lack of good characterizations)

(2) RCEP usually fails in (sub)quasivarieties even if the initial variety has it

Our result

Theorem (Algebra)

There are exactly five subquasivarieties (out of innfinitely! continuum?) of Sugihara algebras with Amalgamation Property. Furthermore, AP$\Longrightarrow$RCEP for Sugihara quasivarieties.

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Theorem (Logical corollary)

There are exactly five consequence relations (not necessarily axiomatic) extendind R–*mingle with Maehara Interpolation Property.*

R – an axiomatic formulation

A1 $p \to p$

A2 $(p \to q) \to ((q \to r) \to (p \to r))$

A3 $p \to ((p \to q) \to q)$

A4 $(p \to (p \to q)) \to (p \to q)$

A5 $p \wedge q \to p$

A6 $p \wedge q \to q$

A7 $((p \to q) \wedge (p \to r)) \to (p \to q \wedge r)$

A8 $p \to p \vee q$

A9 $p \to q \vee p$

A10 $((q \to p) \wedge (r \to p)) \to (q \vee r \to p)$

A11 $p \wedge (q \vee r) \to (p \wedge q) \vee r$

A12 $(p \to \neg q) \to (q \to \neg p)$

A13 $\neg\neg p \to p$

The two rules of the system is modus ponens MP $\{\varphi, \varphi \to \psi\}/\psi$ and the adjunction rule AD $\{\varphi, \psi\}/\varphi \wedge \psi$.

R-mingle and Sugihara algebras

the logic R-mingle results from adding the 'mingle axiom' to basic system of relevance logic R
$p \rightarrow (p \rightarrow p)$.

Sugihara algebras

$\mathrm{Z} = \langle \mathrm{Z}, \wedge, \vee, \rightarrow, \neg \rangle$,

$$x \rightarrow y = \begin{cases} (-x) \vee y & x \leq y \\ (-x) \wedge y & x \not\leq y. \end{cases}$$

Sugihara algebras are memebers of $\mathbb{V}(\mathrm{Z}) = HSP(\mathrm{Z})$.

Subchains (finitely subdirectly irreducibles)

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Sugihara algebras continued

It turns out that the lattice of subvarieties of SA forms a countable chain given by

$$\mathbb{V}(\mathrm{Z}_1) \subseteq \mathbb{V}(\mathrm{Z}_2) \subseteq \mathbb{V}(\mathrm{Z}_3) \subseteq \cdots \mathbb{V}(\mathrm{Z}) = \mathbb{V}(\mathrm{E}) = \mathrm{SA}.$$

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Further, $\mathrm{V}(\mathrm{Z}) = \mathbb{Q}(\mathrm{Z}) = \mathbb{Q}(\{\mathrm{Z}_n \mid n \geq 1\}) = \mathbb{Q}(\{\mathrm{Z}_{2n+1} \mid n \geq 0\})$, and also $\mathbb{Q}(\mathrm{E}) = \mathbb{Q}(\{\mathrm{Z}_{2n} \mid n \geq 1\})$ is a proper subquasivariety of SA.

Positive part

Lemma

Each of the quasivarieties $\mathbb{V}(Z_2)$, $\mathbb{V}(Z_3)$, $\mathbb{V}(Z)$, and $\mathbb{Q}(E)$ has the amalgamation property.

Easy part – all of them have RCEP, so we can apply the known techniques.

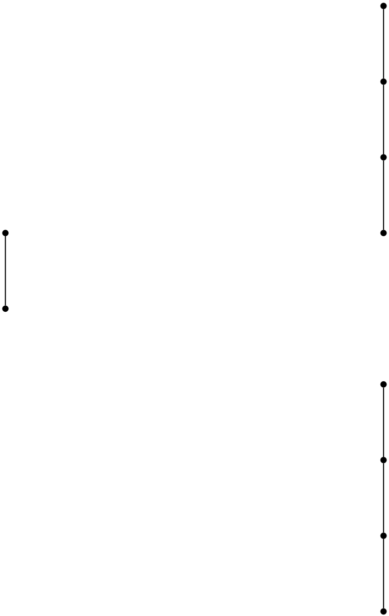
The challenging part– negative part

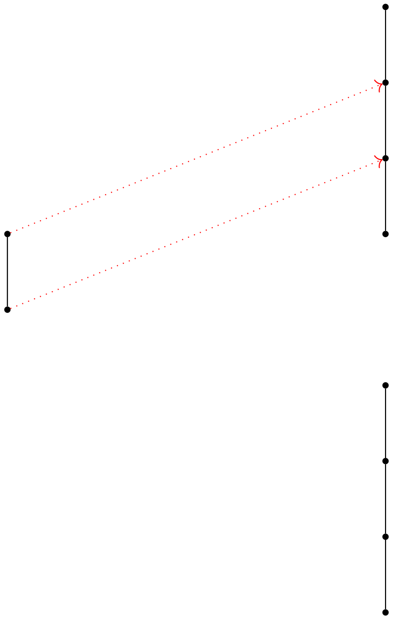
"Any other quasivariety does not have AP"
Our strategy is to use the so-called closure lemmas.

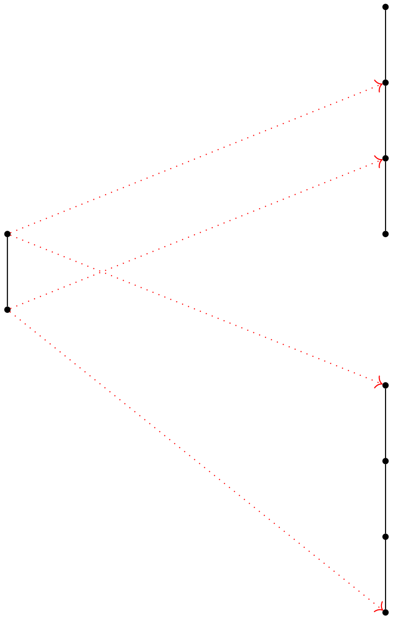
The two extending lemmas (even case)

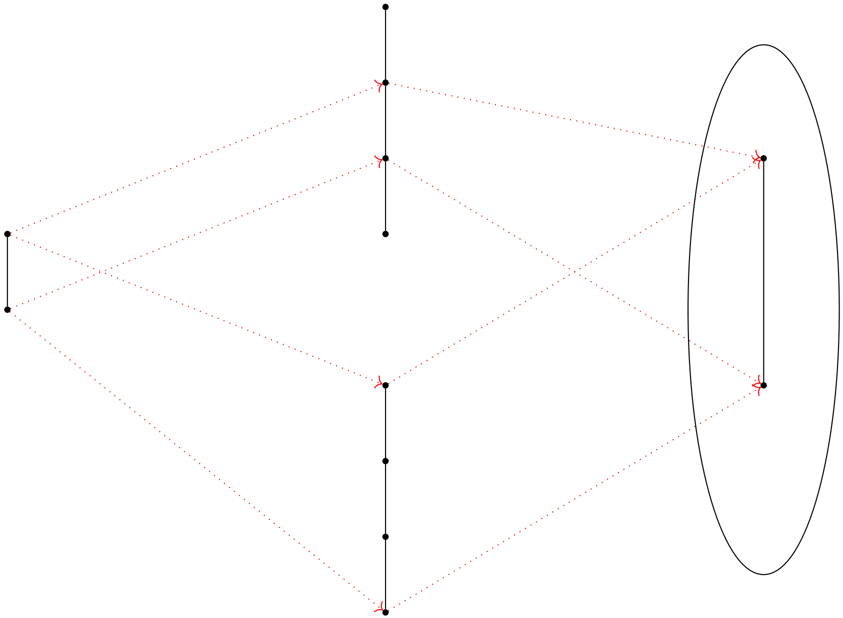
Lemma

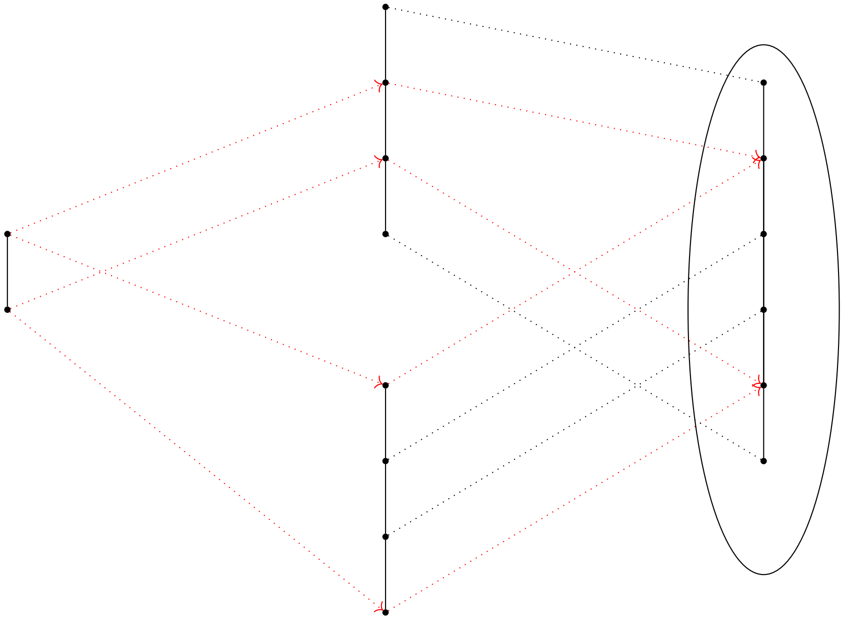
Assume Q *has AP. If* $Z_4 \in Q$, *then* $Z_{2n} \in Q$ *for every positive integer* n.
Consequently, if $Z_4 \in Q$, *then* $E \in Q$.











The two extending lemmas (odd case)

Lemma

Assume Q *has AP. If* $Z_3, Z_4 \in Q$*, then* $Q = SA$.

"The coordinate switch" embedding – odd and even cases

Lemma

Let $\mathbf{Q}$ *has AP. If* $\mathbf{Z}_2 \times \mathbf{Z}_3 \in \mathbf{Q}$*, then* $\mathbf{Z}_3 \in \mathbf{Q}$*.*

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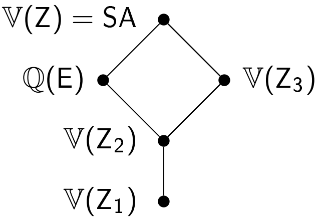
Theorem

Assume a nontrivial subquasivariety of SA, Q has AP. Then it is one of the four: $\mathbb{V}(Z_2)$, $\mathbb{V}(Z_3)$, $\mathbb{V}(Z)$, and $\mathbb{Q}(E)$.

proof idea

Since Q is nontrivial, $\mathbb{V}(Z_2) = \mathbb{Q}(Z_2) \subseteq Q$. Assume that this containment is proper. Then, by Lemma (K K 2022), either $Z_2 \times Z_3 \in Q$ or $Z_2 \times Z_4 \in Q$. We consider three mutually exclusive cases and end up in one of the three remaining quasivarieties: $\mathbb{V}(Z_3)$, $\mathbb{V}(Z)$, or $\mathbb{Q}(E)$

Poset of Qs with AP



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1. In quasivarieties of Sugihara algebras $AP \Longrightarrow RCEP$. (Opposite direction is false) and additionally $AP \Leftrightarrow TIP$ (TIP always implies AP, but not the other way around)
2. In extensions of R–mingle $RP \Leftrightarrow MIP$
3. There is only one non-axiomatic extension of RM (proper quasivariety) which has RP/MIP (AP/TIP).

Thank you!