Paraconsistent Constructive Modal Logic

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Joint work with **Daniil Kozhemiachenko** and **Nicola Olivetti**

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Paraconsistent reasoning and contradictory beliefs

Example

Assume that someone believes that Moon landings did not happen $\Box{\sim}m$ and because of that, there cannot be reflectors on the Moon $\Box({\sim}m \rightarrow {\sim}r)$. On the other hand, they believe that the laser beam emitted towards the Moon is reflected $\Box l$ and that can happen only because there are some reflectors $\Box(l \rightarrow r)$. Thus, they believe in $r \wedge {\sim}r$.

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In the well-known logic N4 in [Nelson, 1949], intuitionistic negation and falsity are replaced by so-called strong negation $\sim$ with all the following features:

1. **constructive falsity**: if ${\sim}(\phi \wedge \chi)$ is provable either ${\sim}\phi$ or ${\sim}\chi$ is provable

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In a paraconsistent setting, contradictory beliefs or obligations are possible.

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How to consider constructive modal logic in a paraconsistent setting?

In this work, we aim to define paraconsistent counterpart(s) of CK with all the features as N4.

Constructive Modal Logic CK

A CK *model* is a quadrupole $\mathfrak{M} = \langle W, \leq, R, V \rangle$ where

- $W \neq \varnothing$, $\leq$ is a preorder on W and R a binary relation
- $V : \mathrm{Prop} \to 2^W$ s.t. $w \in V(p)$ and $w \leq w'$ imply $w' \in V(p)$

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$$\mathfrak{M}, w \Vdash \Box\phi \iff \forall w' \geq w \; \forall w'' \in R(w') : \mathfrak{M}, w'' \Vdash \phi$$
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\mathfrak{M}, w \Vdash \Box\phi \quad &\Longleftrightarrow \quad \forall w' \geq w\ \forall w'' \in R(w') : \mathfrak{M}, w'' \Vdash \phi \\
\mathfrak{M}, w \Vdash \Diamond\phi \quad &\Longleftrightarrow \quad \forall w' \geq w\ \exists w'' \in R(w') : \mathfrak{M}, w'' \Vdash \phi
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Axiomatization of CK

- IPC, Nec
- $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$
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Based on CK and $\sim$, we can get several systems of increasing strength via the relations between the two modal operators $\Box$, $\Diamond$ and their strong-negative counterparts $\sim\Box$, $\sim\Diamond$

Language and semantics of N4

The language of the propositional Nelson's logic (N4) is defined as

$$\mathcal{L}^{\sim} \ni \phi := p \in \mathsf{Prop} \mid {\sim}\phi \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid (\phi \to \phi)$$

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An N4 *model* is a quadrupole $\mathfrak{M} = \langle W, \leq, v^{+}, v^{-} \rangle$ where

- $W \neq \varnothing$ and $\leq$ is a partial preorder on W,
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$$\begin{aligned}
\mathfrak{M}, w \Vdash^{\pm} p &\iff w \in v^{\pm}(p)\\
\mathfrak{M}, w \Vdash^{\pm} {\sim}\phi &\iff \mathfrak{M}, w \Vdash^{\mp} \phi\\
\mathfrak{M}, w \Vdash^{\pm} \phi \wedge \chi &\iff \mathfrak{M}, w \Vdash^{\pm} \phi \text{ and } \mathfrak{M}, w \Vdash^{\pm} \chi\\
\mathfrak{M}, w \Vdash^{\pm} \phi \vee \chi &\iff \mathfrak{M}, w \Vdash^{\pm} \phi \text{ or } \mathfrak{M}, w \Vdash^{\pm} \chi\\
\mathfrak{M}, w \Vdash^{+} \phi \to \chi &\iff \forall w' \geq w : \mathfrak{M}, w' \Vdash^{+} \phi \Rightarrow \mathfrak{M}, w' \Vdash^{+} \chi\\
\mathfrak{M}, w \Vdash^{-} \phi \to \chi &\iff \mathfrak{M}, w \Vdash^{+} \phi \text{ and } \mathfrak{M}, w \Vdash^{-} \chi
\end{aligned}$$

We say that $\phi \in \mathcal{L}^{\sim}$ is N4-*valid* if $\mathfrak{M}, w \Vdash^{+} \phi$ for any N4-model $\mathfrak{M}$ and $w \in \mathfrak{M}$.

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\begin{aligned}
\mathsf{Int}^- &: \text{Axioms for the negation-free intuitionistic logic in } \mathcal{L}^{\sim}. \\
{\sim}{\sim}&: {\sim}{\sim}\phi \leftrightarrow \phi \\
\mathbf{DeM}_\wedge&: {\sim}(\phi \wedge \chi) \leftrightarrow ({\sim}\phi \vee {\sim}\chi) \\
\mathbf{DeM}_\vee&: {\sim}(\phi \vee \chi) \leftrightarrow ({\sim}\phi \wedge {\sim}\chi) \\
\mathbf{DeM}_\to&: {\sim}(\phi \to \chi) \leftrightarrow (\phi \wedge {\sim}\chi) \\
\mathbf{mp}&: \frac{\phi \quad \phi \to \chi}{\chi}
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Int$^-$: Axioms for the negation-free intuitionistic logic in $\mathcal{L}^{\sim}$.

$\sim\sim$: $\sim\sim\phi \leftrightarrow \phi$

DeM$_\wedge$: $\sim(\phi \wedge \chi) \leftrightarrow (\sim\phi \vee \sim\chi)$

DeM$_\vee$: $\sim(\phi \vee \chi) \leftrightarrow (\sim\phi \wedge \sim\chi)$

DeM$_\rightarrow$: $\sim(\phi \rightarrow \chi) \leftrightarrow (\phi \wedge \sim\chi)$

mp: $\dfrac{\phi \quad \phi \rightarrow \chi}{\chi}$

Soundness and completeness

$\mathcal{H}$N4 is sound and complete with respect to the Kripke semantics for N4.

A modal expansion of N4: CN4K

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Definition (CN4K-frames and models)

A CN4K-*frame* is a 6-tuple $\mathfrak{F} = \langle W, \leq, R^{+}_{\Box}, R^{-}_{\Box}, R^{+}_{\Diamond}, R^{-}_{\Diamond} \rangle$ where

- $W, \leq$ defined as in N4-frames
- $R^{\bullet}_{\heartsuit}$ is a binary relation on W for $\bullet \in \{+, -\}$ and $\heartsuit \in \{\Box, \Diamond\}$

A CN4K-*model* is a triple $\mathfrak{M} = \langle \mathfrak{F}, v^{+}, v^{-} \rangle$ where

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A CN4K-frame is called:

$\pm$-*birelational* if $R^+_\Box = R^+_\Diamond$ and $R^-_\Box = R^-_\Diamond$

$\bowtie$-*birelational* if $R^+_\Box = R^-_\Diamond$ and $R^+_\Diamond = R^-_\Box$

$\curlyvee$-*birelational* if $R^+_\Box = R^-_\Box$ and $R^+_\Diamond = R^-_\Diamond$

monorelational if all the four relations coincide

Inpterpreting modalities

Let $\mathfrak{M}$ be a CN4K-model and w a world in it.

$$
\begin{aligned}
\mathfrak{M}, w \Vdash^+ \Box\phi &\iff \forall w' \geq w\ \forall w'' \in R^+_\Box(w') : \mathfrak{M}, w'' \Vdash^+ \phi \\
\mathfrak{M}, w \Vdash^- \Box\phi &\iff \forall w' \geq w\ \exists w'' \in R^-_\Box(w') : \mathfrak{M}, w'' \Vdash^- \phi \\
\mathfrak{M}, w \Vdash^+ \Diamond\phi &\iff \forall w' \geq w\ \exists w'' \in R^+_\Diamond(w') : \mathfrak{M}, w'' \Vdash^+ \phi \\
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We note that $\Box$ and $\Diamond$ are dual to one another via $\sim$:
$\sim\Box\sim$ behaves like $\Diamond$ w.r.t. $R^{-}_{\Box}$ and $\sim\Diamond\sim$ like $\Box$ w.r.t. $R^{-}_{\Diamond}$

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Persistence property

Let $\mathfrak{M}$ be a CN4K-model, $w \leq w'$, and $\phi, \chi \in \mathcal{L}^{\sim}_{\Box,\Diamond}$. Then:

if $\mathfrak{M}, w \Vdash^{+} \phi$, then $\mathfrak{M}, w' \Vdash^{+} \phi$ if $\mathfrak{M}, w \Vdash^{-} \chi$, then $\mathfrak{M}, w' \Vdash^{-} \chi$

Hilbert calculi for CN4K and its extensions

We consider the following CN4K-style logics:

- CN4K: the logic of all CN4K-frames
- $\mathsf{CN4K}^{*}$: the logic of all $*$-birelational frames for $* \in \{\pm, \curlyvee, \bowtie\}$,
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$$\top_\Box := \Box(\phi \to \phi) \qquad \top_\Diamond := {\sim}\Diamond{\sim}(\phi \to \phi)$$

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$$\pm_\Box := \Box(\phi \to \chi) \to (\Diamond\phi \to \Diamond\chi) \qquad \pm_\Diamond := {\sim}\Diamond{\sim}({\sim}\phi \to {\sim}\chi) \to ({\sim}\Box\phi \to {\sim}\Box\chi)$$

$$\curlyvee_\Box := \Box(\phi \to \chi) \to ({\sim}\Box{\sim}\phi \to {\sim}\Box{\sim}\chi) \qquad \curlyvee_\Diamond := {\sim}\Diamond{\sim}(\phi \to \chi) \to (\Diamond\phi \to \Diamond\chi)$$

$$\bowtie_\Box := \Box\phi \leftrightarrow {\sim}\Diamond{\sim}\phi \qquad \bowtie_\Diamond := \Diamond\phi \leftrightarrow {\sim}\Box{\sim}\phi$$

$$\mathbf{r}_\Box := \frac{\vdash \phi \to \chi}{\vdash \Box\phi \to \Box\chi} \quad \mathbf{r}_\Diamond := \frac{\vdash \phi \to \chi}{\vdash \Diamond\phi \to \Diamond\chi} \quad \mathbf{r}^{\sim}_\Box := \frac{\vdash {\sim}\phi \to {\sim}\chi}{\vdash {\sim}\Box\phi \to {\sim}\Box\chi} \quad \mathbf{r}^{\sim}_\Diamond := \frac{\vdash {\sim}\phi \to {\sim}\chi}{\vdash {\sim}\Diamond\phi \to {\sim}\Diamond\chi}$$

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- $\mathcal{H}\mathsf{CN4K} = \mathcal{H}\mathsf{N4} \oplus \{\top_\Box, \top_\Diamond, \wedge_\Box, \wedge_\Diamond, \mathbf{r}_\Box, \mathbf{r}_\Diamond, \mathbf{r}^{\sim}_\Box, \mathbf{r}^{\sim}_\Diamond\}$
- $\mathcal{H}\mathsf{CN4K}^* = \mathcal{H}\mathsf{CN4K} \oplus \{*_\Box, *_\Diamond\}$
- $\mathcal{H}\mathsf{CN4K}^1 = \mathcal{H}\mathsf{CN4K}^\pm \oplus \mathcal{H}\mathsf{CN4K}^\curlyvee \oplus \mathcal{H}\mathsf{CN4K}^\bowtie$

Theorem (Soundness and completeness)

Let $\mathsf{L} \in \{\mathsf{CN4K}, \mathsf{CN4K}^{\pm}, \mathsf{CN4K}^{\curlyvee}, \mathsf{CN4K}^{\bowtie}, \mathsf{CN4K}^{1}\}$*. For* $\Gamma \subseteq_{\mathsf{fin}} \mathcal{L}^{\sim}_{\Box,\Diamond}$ *and* $\phi \in \mathcal{L}^{\sim}_{\Box,\Diamond}$*, we have* $\Gamma \models_{\mathsf{L}} \phi$ *iff* $\Gamma \vdash_{\mathcal{H}\mathsf{L}} \phi$*.*

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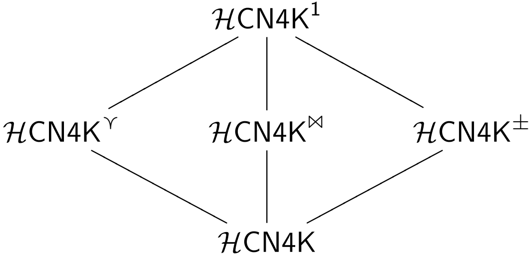

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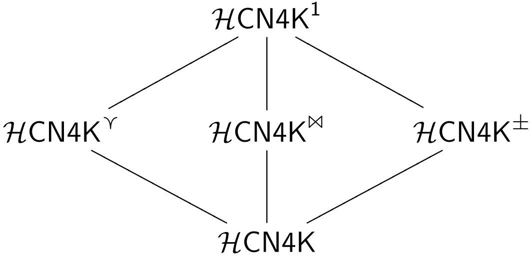
Let $\mathsf{L} \in \{\mathsf{CN4K}, \mathsf{CN4K}^{\pm}, \mathsf{CN4K}^{\curlyvee}, \mathsf{CN4K}^{\bowtie}, \mathsf{CN4K}^{1}\}$*. For* $\Gamma \subseteq_{\mathsf{fin}} \mathcal{L}^{\sim}_{\Box,\Diamond}$ *and* $\phi \in \mathcal{L}^{\sim}_{\Box,\Diamond}$*, we have* $\Gamma \models_{\mathsf{L}} \phi$ *iff* $\Gamma \vdash_{\mathcal{H}\mathsf{L}} \phi$*.*



1. $\mathcal{H}\mathsf{CN4K}^{\pm} \oplus \{\bowtie_{\Box}, \bowtie_{\Diamond}\} \vdash \curlyvee_{\Box} \wedge \curlyvee_{\Diamond}$ and $\mathcal{H}\mathsf{CN4K}^{\curlyvee} \oplus \{\bowtie_{\Box}, \bowtie_{\Diamond}\} \vdash \pm_{\Box} \wedge \pm_{\Diamond}$.
2. $\mathcal{H}\mathsf{CN4K}^{\pm} \oplus \{\curlyvee_{\Box}, \curlyvee_{\Diamond}\} \nvdash \bowtie_{\Box}$ and $\mathcal{H}\mathsf{CN4K}^{\pm} \oplus \{\curlyvee_{\Box}, \curlyvee_{\Diamond}\} \nvdash \bowtie_{\Diamond}$.

Turning to proof theory

A **sequent** is an expression of the form

$$\Gamma \Rightarrow \phi$$

where $\Gamma \sqsubseteq^{\mathsf{fin}} \mathcal{L}^{\sim}_{\Box,\Diamond}$ and $\phi \in \mathcal{L}^{\sim}_{\Box,\Diamond}$.

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▶ We intend to provide complete calculi for all the CN4K-style logics and make use of the calculi to show decidability.

A sequent calculus for N4: $\mathcal{G}$N4

$$\text{Ax}\ \frac{}{p, \Gamma \Rightarrow p} \qquad \text{Ax}_{\sim}\ \frac{}{\sim p, \Gamma \Rightarrow \sim p}$$

$$\wedge_l\ \frac{\phi, \chi, \Gamma \Rightarrow \psi}{\phi \wedge \chi, \Gamma \Rightarrow \psi} \qquad \wedge_r\ \frac{\Gamma \Rightarrow \phi \qquad \Gamma \Rightarrow \chi}{\Gamma \Rightarrow \phi \wedge \chi}$$

$$\vee_l\ \frac{\phi, \Gamma \Rightarrow \psi \qquad \chi, \Gamma \Rightarrow \psi}{\phi \vee \chi, \Gamma \Rightarrow \psi} \qquad \vee_{r_1}\ \frac{\Gamma \Rightarrow \phi}{\Gamma \Rightarrow \phi \vee \chi} \qquad \vee_{r_2}\ \frac{\Gamma \Rightarrow \chi}{\Gamma \Rightarrow \phi \vee \chi}$$

$$\rightarrow_l\ \frac{\phi \rightarrow \chi, \Gamma \Rightarrow \phi \qquad \chi, \Gamma \Rightarrow \psi}{\phi \rightarrow \chi, \Gamma \Rightarrow \psi} \qquad \rightarrow_r\ \frac{\phi, \Gamma \Rightarrow \chi}{\Gamma \Rightarrow \phi \rightarrow \chi}$$

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$$\sim\vee_l\ \frac{\sim\phi, \sim\chi, \Gamma \Rightarrow \psi}{\sim(\phi \vee \chi), \Gamma \Rightarrow \psi} \qquad \sim\vee_r\ \frac{\Gamma \Rightarrow \sim\phi \qquad \Gamma \Rightarrow \sim\chi}{\Gamma \Rightarrow \sim(\phi \vee \chi)}$$

$$\sim\wedge_l\ \frac{\sim\phi, \Gamma \Rightarrow \psi \qquad \sim\chi, \Gamma \Rightarrow \psi}{\sim(\phi \vee \chi), \Gamma \Rightarrow \psi} \qquad \sim\wedge_{r_1}\ \frac{\Gamma \Rightarrow \sim\phi}{\Gamma \Rightarrow \sim(\phi \wedge \chi)} \qquad \sim\wedge_{r_2}\ \frac{\Gamma \Rightarrow \sim\chi}{\Gamma \Rightarrow \sim(\phi \wedge \chi)}$$

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Figure: Rules of $\mathcal{G}$N4, adapted from [Kamide and Wansing, 2012].

Sequent calculi for CN4K-logics

Let $\Gamma^{\Box} = \{\phi \mid \Box\phi \in \Gamma\}$ and $\Gamma^{\Diamond}_{\sim} = \{\sim\phi \mid \sim\Diamond\phi \in \Gamma\}$ where Γ is a multiset of formulas.

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We define the following calculi:

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We define the following calculi:

$$\mathcal{G}\mathsf{CN4K} = \mathcal{G}\mathsf{N4} \oplus \{\Box, \Diamond, \Box_{\sim}, \Diamond_{\sim}\} \quad \mathcal{G}\mathsf{CN4K}^{\curlyvee} = \mathcal{G}\mathsf{N4} \oplus \{\Box, \Diamond^{\curlyvee}, \Box^{\curlyvee}_{\sim}, \Diamond_{\sim}\}$$

$$\mathcal{G}\mathsf{CN4K}^{\pm} = \mathcal{G}\mathsf{N4} \oplus \{\Box, \Diamond^{\pm}, \Box^{\pm}_{\sim}, \Diamond_{\sim}\} \quad \mathcal{G}\mathsf{CN4K}^{\bowtie} = \mathcal{G}\mathsf{N4} \oplus \{\Box^{\bowtie}, \Diamond, \Diamond^{\bowtie}, \Box_{\sim}, \Box^{\bowtie}_{\sim}, \Diamond^{\bowtie}_{\sim}\}$$

$$\mathcal{G}\mathsf{CN4K}^{1} = \mathcal{G}\mathsf{N4} \oplus \{\Box^{\bowtie}, \Diamond^{1}, \Diamond^{1,\bowtie}, \Box^{1}_{\sim}, \Box^{1,\bowtie}_{\sim}, \Diamond^{\bowtie}_{\sim}\}$$

Example

Proposition

For any formula ϕ and Γ, sequent $\Gamma, \phi \Rightarrow \phi$ is provable in $\mathcal{G}$L.

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We show that axiom $\bowtie_\Box$ is provable in $\mathcal{G}$CN4K$^{\bowtie}$.

$$
\dfrac{
\dfrac{\dfrac{\dfrac{\overline{\phi \Rightarrow \phi}}{\phi \Rightarrow {\sim}{\sim}\phi}\,{\sim}{\sim}_r}{\Box\phi \Rightarrow {\sim}\Diamond{\sim}\phi}\,\Diamond^{\bowtie}_{\sim}}{\Rightarrow \Box\phi \to {\sim}\Diamond{\sim}\phi}\,{\to}_r
\qquad
\dfrac{\dfrac{\dfrac{\overline{\phi \Rightarrow \phi}}{{\sim}{\sim}\phi \Rightarrow \phi}\,{\sim}{\sim}_l}{{\sim}\Diamond{\sim}\phi \Rightarrow \Box\phi}\,\Box^{\bowtie}}{\Rightarrow {\sim}\Diamond{\sim}\phi \to \Box\phi}\,{\to}_r
}{\Rightarrow (\Box\phi \to {\sim}\Diamond{\sim}\phi) \wedge ({\sim}\Diamond{\sim}\phi \to \Box\phi)}\,\wedge_r
$$

Cut-admissibility

The following rules are height-preserving admissible in $\mathcal{G}\mathsf{L}$:

$$\text{weakening}\ \frac{\Gamma \Rightarrow \chi}{\phi, \Gamma \Rightarrow \chi} \qquad \text{contraction}\ \frac{\phi, \phi, \Gamma \Rightarrow \chi}{\phi, \Gamma \Rightarrow \chi}$$

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Further results

Completeness: if $\mathsf{L} \models \bigwedge \Gamma \rightarrow \phi$, then $\mathcal{G}\mathsf{L} \vdash \Gamma \Rightarrow \phi$.

Decidability: validity in CN4K and its extensions is decidable.

To sum up

In this work, we have obtained the following results:

- proposed a family of paraconsistent counterparts of the constructive modal logic CK
- provided sound and complete Hilbert-style axiomatization
- presented modular cut-free sequent calculi
- showed cut-admissibility and then decidability via the calculi

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In this work, we have obtained the following results:

- proposed a family of paraconsistent counterparts of the constructive modal logic CK
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Topics for future research:

- complexity of the decision problem: PSPACE?
- calculi supporting countermodel construction
- similar logics with specific frame conditions between $\leq$ and $R^{\pm}$
- extensions defined by properties of the accessibility relations that characterize the epistemic, doxastic, or deontic interpretation of the modalities

Thanks for your attention!

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Paraconsistent Constructive Modal Logic

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Czech Gathering of Logicians & Beauty of Logic @ Prague, 4th Feb, 2026