Hybrid logic of strict betweenness

Rafał Gruszczyński and Zhiguang Zhao

Department of Logic
Nicolaus Copernicus University in Toruń
Poland

School of Mathematics and Statistics
Taishan University in Tai'an
China

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Outline

The idea

van Benthem and Bezhanishvili (2007): a ternary betweenness relation B gives rise to the binary modal operator $\langle B \rangle$ whose relational semantics is given by the following condition:

$$x \Vdash \langle B \rangle (\varphi, \psi) \;:\longleftrightarrow\; (\exists\, y, z \in U)\,(B(y, x, z) \text{ and } y \Vdash \varphi \text{ and } z \Vdash \psi)\,,$$

where $B(y, x, z)$ is interpreted as

point x is between points y and z.

The goal: study the operator in the setting of hybrid logic.

Outline

First-order axioms

For the basic characterization of (strict) betweenness, we are going to use the following standard set of first-order axioms:

$$B(x, y, z) \to \#(x, y, z)\,, \tag{B1}$$

$$B(x, y, z) \to B(z, y, x)\,, \tag{B2}$$

$$B(x, y, z) \to \neg B(x, z, y)\,, \tag{B3}$$

$$B(x, y, z) \wedge B(y, z, u) \to B(x, y, u)\,, \tag{B4}$$

$$B(x, y, z) \wedge B(y, u, z) \to B(x, y, u)\,, \tag{B5}$$

$$\#(x, y, z) \to B(x, y, z) \vee B(x, z, y) \vee B(y, x, z)\,, \tag{B6}$$

$$\forall y \exists x \exists z\, B(x, y, z)\,, \tag{B7}$$

$$x \neq z \to \exists y\, B(x, y, z)\,. \tag{B8}$$

The source: (Borsuk and Szmielew, 1960).

First-order axioms

$$B(x,y,z) \to \#(x,y,z)\,, \tag{B1}$$

$$B(x,y,z) \to B(z,y,x)\,, \tag{B2}$$

$$B(x,y,z) \to \neg B(x,z,y)\,, \tag{B3}$$

$$B(x,y,z) \land B(y,z,u) \to B(x,y,u)\,, \tag{B4}$$

$$B(x,y,z) \land B(y,u,z) \to B(x,y,u)\,, \tag{B5}$$

$$\#(x,y,z) \to B(x,y,z) \lor B(x,z,y) \lor B(y,x,z)\,, \tag{B6}$$

$$\forall y \exists x \exists z\, B(x,y,z)\,, \tag{B7}$$

$$x \neq z \to \exists y\, B(x,y,z)\,. \tag{B8}$$

Definition

Any 3-frame $\mathfrak{F}$ that satisfies (B1)–(B7) will be called a linear betweenness frame without endpoints. The class of all such frames will be denoted by **LBWE.** The elements of the class

$$\mathbf{DLBWE} := \mathbf{LBWE} + \text{(B8)}$$

will be called dense linear betweenness frames without endpoints.

Outline

Basic modal language

As the betweenness relation is ternary, for its modal analysis we will need a binary modal operator $\langle B\rangle$. Let β be the modal similarity type with $\langle B\rangle$ as the only operator. The modal language $\mathcal{L}_\beta$ is given by the following definition

$$\varphi \coloneqq p \mid \top \mid \neg\varphi \mid \varphi \wedge \psi \mid \langle B\rangle(\varphi, \psi)\,.$$

Semantics

Given a model $\mathfrak{M} := \langle \mathfrak{F}, V \rangle$ based on a 3-frame $\mathfrak{F}$ we characterize the semantic for $\langle B \rangle$ in the following way

$$\begin{aligned}\mathfrak{M}, w \Vdash \langle B \rangle(\varphi, \psi) \;&:\Longleftrightarrow \\ &(\exists x, y \in W)\,(\mathfrak{M}, x \Vdash \varphi \text{ and } \mathfrak{M}, y \Vdash \psi \text{ and } B(x, w, y))\,.\end{aligned} \tag{$\mathtt{df}\,\langle B \rangle$}$$

$\langle B \rangle$ gives rise to a unary convexity operator

$$\mathsf{C}\,\varphi \;:\Longleftrightarrow\; \langle B \rangle(\varphi, \varphi)\,. \tag{$\mathtt{df}\,\mathsf{C}$}$$

The name of the operator is motivated by the fact that it is closely related to the operation of the convex closure of a set. For details, see (van Benthem and Bezhanishvili, 2007, pp. 281–282).

Definability in $\mathcal{L}_\beta$

$$B(x, y, z) \to B(z, y, x)\,, \tag{B2}$$

$$\forall y \exists x \exists z\, B(x, y, z)\,. \tag{B7}$$

Theorem

Any 3-frame $\mathfrak{F}$ satisfies (the universal closure of) (B2) *iff the following formula is valid in $\mathfrak{F}$*

$$\langle B\rangle(\varphi, \psi) \to \langle B\rangle(\psi, \varphi)\,.$$

Theorem

Any 3-frame $\mathfrak{F}$ satisfies (the universal closure of) (B7) *iff $\langle B\rangle(\top, \top)$ is valid in $\mathfrak{F}$.*

Non-definability in $\mathcal{L}_\beta$

$$B(x, y, z) \to \#(x, y, z)\,, \tag{B1}$$

$$B(x, y, z) \to \neg B(x, z, y)\,, \tag{B3}$$

$$B(x, y, z) \wedge B(y, z, u) \to B(x, y, u)\,, \tag{B4}$$

$$B(x, y, z) \wedge B(y, u, z) \to B(x, y, u)\,, \tag{B5}$$

$$\#(x, y, z) \to B(x, y, z) \vee B(x, z, y) \vee B(y, x, z)\,. \tag{B6}$$

$$x \neq z \to \exists y\, B(x, y, z)\,. \tag{B8}$$

Theorem

For every $i \in \{1, 3, 4, 5, 6\}$ the class of (Bi)*-frames is not $\mathcal{L}_\beta$-definable.*

Hybrid language

The basic hybrid language $\mathcal{H}_\beta$ is a two-sorted language that is an expansion of $\mathcal{L}_\beta$ with nominals i, j, k, l (indexed if necessary)

$$\varphi := p \mid \top \mid i \mid \neg\varphi \mid \varphi \wedge \psi \mid \langle B\rangle(\varphi, \psi)\,.$$

The set of all propositional letters will be denoted by 'Prop', and the set of nominals by 'Ω'. We assume that $\mathrm{Prop} \cap \Omega = \emptyset$.

The hybrid language that plays the most important role in the sequel is $\mathcal{H}_\beta(@)$ that expands $\mathcal{H}_\beta$ with a family of satisfaction operators $@_i$, one for every nominal

$$\varphi := p \mid \top \mid i \mid \neg\varphi \mid \varphi \wedge \psi \mid \langle B\rangle(\varphi, \psi) \mid @_i\, \varphi\,.$$

Hybrid language: semantics

Definition

Given a frame $\mathfrak{F} := \langle W, B \rangle$, a valuation function is any function $V\colon \mathrm{Prop} \cup \Omega \to 2^W$ such that for every nominal i, $V(i)$ is a singleton subset of the universe.

So

$$\mathfrak{M}, w \Vdash i \iff V(i) = \{w\}\,.$$

The semantics of the at operator is given by the following

$$\mathfrak{M}, w \Vdash @_i \varphi \;:\Longleftrightarrow\; \mathfrak{M}, V(i) \Vdash \varphi\,. \qquad (\mathtt{df}\ @_i)$$

For the betweenness modality we have

$$\begin{aligned}\mathfrak{M}, w \Vdash \langle B \rangle(i,j) &\iff (\exists x, y \in W)\,(V(i) = \{x\} \text{ and } V(j) = \{y\} \text{ and } B(x,w,y))\\ &\iff B(V(i), w, V(j)).\end{aligned}$$

Definability criteria for $\mathcal{H}_\beta(@)$

Definition

A frame $\mathfrak{G}$ is an ultrafilter bounded morphic image of $\mathfrak{F}$ iff there is a surjective bounded morphism $f\colon \mathfrak{F} \to \mathfrak{ue}\,\mathfrak{G}$ such that for every principal ultrafilter $\mathscr{U}$ in $\mathfrak{ue}\,\mathfrak{G}$ it is the case that $|f^{-1}(\{\mathscr{U}\})| = 1$.[1]

Theorem

Let τ be a modal similarity type, and $\mathcal{H}_\tau(@)$ be a hybrid language built with τ. A first-order definable class of Kripke τ-frames is $\mathcal{H}_\tau(@)$-definable by arbitrary formulas iff it is closed under taking ultrafilter bounded morphic images and generated subframes.

Originally proven in (ten Cate, 2005) for the unary case, generalized to arbitrary arities by (Gruszczyński and Zhao, 2025).

[1] The condition $|f^{-1}(\{\mathscr{U}\})| = 1$ says that f is injective w.r.t. principal ultrafilters, and the idea is that such ultrafilters correspond to nominals.

Non-definability in $\mathcal{H}_\beta(@)$

$$x \neq z \rightarrow \exists y\, B(x, y, z) \tag{B8}$$

Theorem

The class of (B8)*-frames is not* $\mathcal{H}_\beta(@)$*-definable*

Proof.

(1) Consider the frame $\mathfrak{F} := \langle [0,1]^{\mathbb{Q}}, B_< \rangle$ where $[0,1]^{\mathbb{Q}}$ is a closed interval of rational numbers and $B_<$ is induced by the irreflexive order $<$ on the interval.

(2) The subframe $\mathfrak{F}' := \langle \{0,1\}, \emptyset \rangle$ is generated, since neither 0 nor 1 are between any numbers from the interval.

(3) But $\mathfrak{F}'$ fails to meet density in an obvious way.

□

Non-definability of **DLBWE**

Proposition

DLBWE *is not* $\mathcal{L}_\beta$*-definable.*

Proof.

If $\mathfrak{F} \in$ **DLBWE**, then the unique mapping onto the frame

$$\mathfrak{F}' := \langle \{x\}, \{\langle x, x, x \rangle\} \rangle$$

is a bounded morphism. Thus the class is not closed for bounded morphic images. □

Outline

Pure axioms for betweenness

$$B(x,y,z) \to \#(x,y,z)\,, \tag{B1}$$
$$B(x,y,z) \to \neg B(z,y,x)\,, \tag{B2}$$
$$B(x,y,z) \to \neg B(x,z,y)\,, \tag{B3}$$
$$B(x,y,z) \wedge B(y,z,u) \to B(x,y,u)\,, \tag{B4}$$
$$B(x,y,z) \wedge B(y,u,z) \to B(x,y,u)\,, \tag{B5}$$
$$\#(x,y,z) \to B(x,y,z) \vee B(x,z,y) \vee B(y,x,z)\,, \tag{B6}$$
$$\forall x \exists y \exists z\, B(x,y,z)\,. \tag{B7}$$

$$@_j \langle B\rangle(i,k) \to \neg\, @_j\, i \wedge \neg\, @_j\, k \wedge \neg\, @_k\, i\,, \tag{HB1}$$
$$\langle B\rangle(i,j) \to \langle B\rangle(j,i)\,, \tag{HB2}$$
$$@_j \langle B\rangle(i,k) \to \neg\, @_k \langle B\rangle(i,j)\,, \tag{HB3}$$
$$@_j \langle B\rangle(i,k) \wedge @_k \langle B\rangle(j,l) \to @_j \langle B\rangle(i,l) \wedge @_k \langle B\rangle(i,l)\,, \tag{HB4}$$
$$@_j \langle B\rangle(i,k) \wedge @_l \langle B\rangle(i,j) \to @_l \langle B\rangle(i,k) \wedge @_j \langle B\rangle(l,k)\,, \tag{HB5}$$
$$\neg\, @_i\, j \wedge \neg\, @_i\, k \wedge \neg\, @_k\, j \to @_j \langle B\rangle(i,k) \vee @_k \langle B\rangle(i,j) \vee @_i \langle B\rangle(j,k)\,, \tag{HB6}$$
$$\langle B\rangle(\top,\top)\,. \tag{HB7}$$

Pure axioms form betweenness: density

As for the density axiom, its corresponding class of frames is not $\mathcal{H}_\beta(@)$-definable. However, it can be captured directly by means of a formula of $\mathcal{H}_\beta(@, \mathsf{E})$-language, where E is the existential modality

$$\neg @_i j \to \mathsf{E}\,\langle B\rangle(i,j)\,. \tag{HB8}$$

Theorem

For any frame $\mathfrak{F} \in$ **LBWE**: $\mathfrak{F} \models$ (B8) *iff* $\mathfrak{F} \Vdash \mathsf{C}\,p \to \mathsf{C}\,\mathsf{C}\,p$.[2]

[2] The idea for the theorem comes from (Bezhanishvili et al., 2023).

Pure axioms form betweenness: density

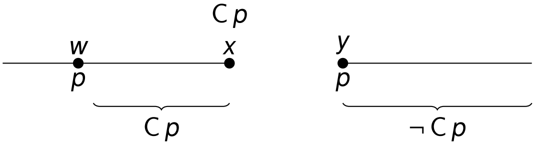

Figure: Failure of the modal density axiom in a non-dense frame. Since there is no point between x and y, $x \nVdash \mathrm{C}\,\mathrm{C}\,p$

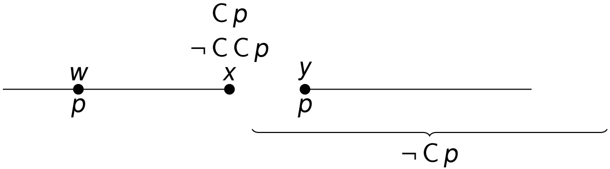

Figure: Failure of the modal density axiom entails failure of the geometrical density. The diagram above presents the situation in which all points on the y side of x fail to satisfy $\mathrm{C}\,p$.

Pure axioms form betweenness: density

The missing piece for pure axiomatization of the strict betweenness:

Theorem

For any frame $\mathfrak{F} \in \mathbf{LBWE}$: $\mathfrak{F} \models$ (B8) *iff* $\mathfrak{F} \Vdash \langle B \rangle(i, j) \rightarrow \mathsf{C}\,\mathsf{C}(i \vee j)$.

$$\langle B \rangle(i, j) \rightarrow \mathsf{C}\,\mathsf{C}(i \vee j) \tag{HB8$'$}$$

Outline

Modal axioms

All classical propositional tautologies (CT)

$$\neg \langle B\rangle(p,q) \leftrightarrow [B](\neg p, \neg q) \qquad \text{(dual)}$$

$$[B](p \rightarrow q, r) \rightarrow ([B](p,r) \rightarrow [B](q,r)) \qquad (K_1)$$

$$[B](r, p \rightarrow q) \rightarrow ([B](r,p) \rightarrow [B](r,q)) \qquad (K_2)$$

Hybrid axioms

$$@_i(p \to q) \to (@_i\, p \to @_i\, q) \qquad (\mathtt{K}_{@})$$

$$\neg\, @_i\, p \leftrightarrow @_i \neg p \qquad (\mathtt{selfdual})$$

$$@_i\, i \qquad (\mathtt{ref})$$

$$i \wedge p \to @_i\, p \qquad (\mathtt{intro})$$

$$\langle B \rangle(@_i\, p, q) \to @_i\, p \qquad (\mathtt{back}_1)$$

$$\langle B \rangle(q, @_i\, p) \to @_i\, p \qquad (\mathtt{back}_2)$$

$$@_i\, @_j\, p \to @_j\, p \qquad (\mathtt{agree})$$

$$@_j\, i \to @_i\, j \qquad (\mathtt{sym})$$

$$@_i\, j \wedge @_j\, p \to @_i\, p \qquad (\mathtt{nom})$$

Rules

$$\text{From } \vdash \varphi \to \psi \text{ and } \vdash \varphi \text{ infer } \vdash \psi. \tag{MP}$$

$$\text{From } \vdash \varphi \text{ infer } \vdash [B](\varphi, \psi). \tag{Nec$_1$}$$

$$\text{From } \vdash \psi \text{ infer } \vdash [B](\varphi, \psi). \tag{Nec$_2$}$$

$$\text{From } \vdash \varphi \text{ infer } \vdash @_i \varphi \tag{Nec$_@$}$$

From $\vdash \varphi$ infer $\vdash \sigma(\varphi)$ where $\sigma(\varphi)$ is a uniform substitution which replaces propositional variables by formulas and nominals by nominals. (Subst)

From $\vdash i \to \theta$ infer $\vdash \theta$, if i does not occur in θ. (Name)

From $\vdash @_i \langle B \rangle (j, k) \wedge @_j \varphi \wedge @_k \psi \to \theta$ infer $\vdash @_i \langle B \rangle (\varphi, \psi) \to \theta$, if i, j, k are pairwise different and j, k do not occur in φ, ψ and θ. (Paste)

The system $\mathbf{B}_h$

Definition

$\mathbf{B}_h$ is a system of hybrid logic determined by the above-mention axioms and rules, plus the specific axioms $(\mathtt{HB1})$–$(\mathtt{HB7})$ and $(\mathtt{HB8}')$.

The system $\mathbf{B}_h$: completeness

Lemma (Blackburn et al., 2001, Lemma 7.22)

Given a named model $\mathfrak{M} := \langle \mathfrak{F}, V \rangle$ *and a pure formula* φ*, if* $\mathfrak{M} \Vdash \psi$ *for all pure instances* ψ *of* φ*, then* $\mathfrak{F} \Vdash \varphi$*.*

Theorem

Every $\mathbf{B}_h$*-consistent set of formulas is satisfiable in a countable named model based on a frame that validates every specific axiom of logic* $\mathbf{B}_h$*, i.e.,* $(\mathtt{HB1})$–$(\mathtt{HB7})$ *and* $(\mathtt{HB8'})$*. Hence, every* $\mathbf{B}_h$*-consistent set of formulas is satisfiable on a model based on a countable frame from the class* **DLBWE***.*

Corollary

Every $\mathcal{H}_\beta(@)$*-formula valid on the frame* $\mathfrak{Q} = \langle \mathbb{Q}, B_< \rangle$ *is a theorem of* $\mathbf{B}_h$*.*

Dedekind completeness

We now move on to the subclass of **DLBWE** that contains only complete structures, i.e., those which satisfy the following second-order axiom

$$(\forall X, Y \in 2^W)\,[(\exists w \in W)(\forall x \in X)(\forall y \in Y)\, B(w,x,y) \rightarrow (\exists u \in W)(\forall x \in X)(\forall y \in Y)\, B(x,u,y)]\,. \tag{B9}$$

We define

$$\textbf{CDLBWE} := \textbf{DLBWE} + (\text{B9})\,,$$

the class of Dedekind complete (D-complete).

Dedekind completeness

- Being second-order, $(\mathtt{B9})$ axiom cannot have a pure counterpart formula built in $\mathcal{H}_\beta(@, \mathsf{E})$.
- Moreover, since every structure from **CDLBWE** has an uncountable domain, the class cannot be axiomatized by means of first-order formulas (by Löwenheim-Skolem theorem).

In consequence,

Theorem

The class **CDLBWE** *is not* $\mathcal{H}_\beta(@, \mathsf{E})$*-definable by pure formulas.*

Dedekind completeness: modal form

Theorem

For any frame $\mathfrak{F} := \langle W, B \rangle$, *any valuation* V

$$\langle \mathfrak{F}, V \rangle \Vdash \mathsf{C}\, p \to p \quad \textit{if and only if} \quad V(p) \textit{ is convex.}$$

Dedekind completeness: modal form

Let $\mathsf{A} := \neg\,\mathsf{E}\,\neg$. The modal Dedekind completeness axiom is the following formula

$$\begin{aligned}\mathsf{E}\,\mathsf{C}\,p \wedge \mathsf{E}\,\mathsf{C}\,q \wedge \mathsf{A}(\mathsf{C}\,p \to p) \wedge \mathsf{A}(\mathsf{C}\,q \to q) \wedge \neg\,\mathsf{E}(p \wedge q) \to \\ \mathsf{E}(\langle B\rangle(p,q) \wedge \neg\,\mathsf{C}\,p \wedge \neg\,\mathsf{C}\,q)\,.\end{aligned} \tag{D}$$

The axiom embodies the following intuition.

- Suppose there exists a pair of distinct points satisfying p and a pair of distinct points satisfying q (this is postulated by $\mathsf{E}\,\mathsf{C}\,p \wedge \mathsf{E}\,\mathsf{C}\,q$).
- Since all points between any pair of p-points satisfy $\mathsf{C}\,p$, $\mathsf{A}(\mathsf{C}\,p \to p)$ postulates that the set of all p-points is an interval (i.e., a convex set). Analogously, the set of all q-points is an interval, too.
- By $\neg\,\mathsf{E}(p \wedge q)$, the two intervals are disjoint.
- Thus, the antecedent of (D) says: there is a pair of non-empty disjoint intervals (so one of them being entirely to the one side of another).
- Then, there is a point that is located between a p-point and a q-point, but not between any pair of p-points, nor any pair of q-points. Briefly, there is a point between the two intervals from the antecedent.

Dedekind completeness: modal form

$$\mathsf{E}\,\mathsf{C}\,p \wedge \mathsf{E}\,\mathsf{C}\,q \wedge \mathsf{A}(\mathsf{C}\,p \to p) \wedge \mathsf{A}(\mathsf{C}\,q \to q) \wedge \neg\,\mathsf{E}(p \wedge q) \to \\ \mathsf{E}(\langle B\rangle(p,q) \wedge \neg\,\mathsf{C}\,p \wedge \neg\,\mathsf{C}\,q)\,. \tag{D}$$

Theorem

If $\mathfrak{F} \in \mathbf{LBWE}$, *then* $\mathfrak{F} \Vdash \mathsf{E}\,\varphi \leftrightarrow \langle B\rangle(\varphi, \top) \vee \varphi$.

Dedekind completeness: modal form

$$\begin{aligned}\mathsf{E}\,\mathsf{C}\,p \wedge \mathsf{E}\,\mathsf{C}\,q \wedge \mathsf{A}(\mathsf{C}\,p \to p) \wedge \mathsf{A}(\mathsf{C}\,q \to q) \wedge \neg\,\mathsf{E}(p \wedge q) \to \\ \mathsf{E}(\langle B\rangle(p,q) \wedge \neg\,\mathsf{C}\,p \wedge \neg\,\mathsf{C}\,q)\,.\end{aligned} \tag{D}$$

Lemma

(D) *is valid in the class* **CDLBWE**.

The system $\mathbf{B}_h^+$

$$\mathsf{E}\,\mathsf{C}\,p \wedge \mathsf{E}\,\mathsf{C}\,q \wedge \mathsf{A}(\mathsf{C}\,p \to p) \wedge \mathsf{A}(\mathsf{C}\,q \to q) \wedge \neg\,\mathsf{E}(p \wedge q) \to \mathsf{E}(\langle B \rangle(p, q) \wedge \neg\,\mathsf{C}\,p \wedge \neg\,\mathsf{C}\,q)\,. \tag{D}$$

Definition

$\mathbf{B}_h^+$ is logic $\mathbf{B}_h$ extended with the modal Dedekind completeness axiom $(\mathtt{D})$.

The system $\mathbf{B}_h^+$: completeness

Two observations:

- For a fixed $\mathbf{B}_h^+$-consistent set of sentences Γ, the canonical model $\mathfrak{M}^\Gamma$ yielded by Γ satisfies all the axioms of $\mathbf{B}_h$, and thus the canonical frame $\mathfrak{F}^\Gamma$—on which $\mathfrak{M}^\Gamma$ is based—validates the axioms (which are pure formulas). Since the frame being named is countable, it is isomorphic to $\mathfrak{Q} = \langle \mathbb{Q}, B_< \rangle$. $(\mathtt{D})$ is of course globally true in $\mathfrak{M}^\Gamma$, yet it cannot be valid on $\mathfrak{F}^\Gamma$, as otherwise $\mathfrak{F}^\Gamma$, and so $\mathfrak{Q}$, would be D-complete.
- Thus, to show completeness of $\mathbf{B}_h^+$ with respect to **CDLBWE** we need to expand the domain of $\mathfrak{F}^\Gamma$ with new elements that will guarantee D-completeness. However, the expansion must be made in such a way that betweenness holds among new points, and the truth lemma is preserved for the new model. So in particular, the truth lemma for the old "rational" points is not affected by the expansion.

The system $\mathbf{B}_h^+$: completeness

Definition

If x is an irrational number and $(a, b) \subseteq \mathbb{Q}$, then $x \lessdot (a, b)$ if x is an element of the standard, minimal completion of (a, b).

Definition

Given a rational based model $\mathfrak{M} := \langle \mathfrak{Q}, V \rangle$ and an irrational number x, we call a formula ϕ local at x, if there are $a, a' \in \mathbb{Q}$ such that $x \lessdot (a, a')$ and $(a, a') \subseteq V(\phi)$.

The idea is that ϕ is local at x if there is an interval $(a, a') \subseteq \mathbb{Q}$ such that ϕ is satisfied at every point of the interval. In principle, we want that if all the rational points that are in the vicinity of x make ϕ true, then ϕ should also be true at x.

The system $\mathbf{B}_h^+$: completeness

For a fixed irrational number x, let us define the following three sets:

$$\begin{aligned}
\Lambda_x &:= \{\phi \mid \phi \text{ is local at } x\}, \\
\Sigma_x &:= \{\langle B\rangle(\phi,\psi) \mid \text{there are } a, a' \text{ such that } a \Vdash \phi, a' \Vdash \psi \text{ and } x \lessdot (a,a')\}, \\
\Pi_x &:= \{\neg\langle B\rangle(\phi,\psi) \mid \langle B\rangle(\phi,\psi) \notin \Sigma_x\}.
\end{aligned}$$

Observe that $\Sigma_x \subseteq \Lambda_x$.

Theorem

For any irrational number x, $\Lambda_x \cup \Pi_x$ is $\mathbf{B}_h^+$-consistent, and so it has a maximally consistent extension Γ_x.

The system $\mathbf{B}_h^+$: completeness

Let Γ be a $\mathbf{B}_h^+$-consistent set of sentences. The named model $\mathfrak{M}^\Gamma$ yielded by Γ is countable, and all axioms of $\mathbf{B}_h$ are globally true in it. Therefore, $\mathfrak{F}^\Gamma$ is isomorphic to $\mathfrak{Q}$. We expand $\mathfrak{F}^\Gamma$ to the frame $\mathfrak{F}_+^\Gamma$ such that

(i) the domain W_+^Γ of $\mathfrak{F}_+^\Gamma$ is W^Γ extended with a copy of Γ_x for each irrational number x,

(ii) $B_+^\Gamma(\Gamma_x, \Gamma_y, \Gamma_z) \;:\Longleftrightarrow\; x < y < z$ or $z < x < y$ (for all reals x, y, z),

and we turn the frame into a model $\mathfrak{M}_+^\Gamma := \langle \mathfrak{F}_+^\Gamma, V_+^\Gamma \rangle$ in the standard way,

$$\Gamma_x \in V_+^\Gamma(p) \;:\Longleftrightarrow\; p \in \Gamma_x \,.$$

For nominals, $V_+^\Gamma(i) = V^\Gamma(i)$, as these can only be true at rational indexed sets

$$\Gamma_x \in V_+^\Gamma(i) \Longleftrightarrow \Gamma_x \in V^\Gamma(i) \Longleftrightarrow x \in \mathbb{Q} \text{ and } i \in \Gamma_x \,.$$

The system $\mathbf{B}_h^+$: completeness

Theorem (Completeness Theorem for $\mathbf{B}_h^+$)

*If Γ is a $\mathbf{B}_h^+$-consistent set of formulas, then there is a model of Γ that is based on a frame $\mathfrak{F} \in$ **CDLBWE** such that $\mathfrak{F}$ is isomoprhic to $\langle \mathbb{R}, B_< \rangle$.*

Corollary

Every $\mathcal{H}_\beta(@)$-formula valid on the frame $\langle \mathbb{R}, B_< \rangle$ is a theorem of $\mathbf{B}_h^+$.

A negative result about $\mathcal{H}_\beta(@, \mathsf{E})$

- As is well-known, the real line $\langle \mathbb{R}, < \rangle$ can be characterized up-to-isomorphism by the following conditions: it is a complete, dense linear order without endpoints which is separable, i.e., has a countable subset D such every non-empty open interval has a common point with D.
- In light of mutual equivalence between strict orders and strict betweenness, this characterization can be also applied to $\mathfrak{R} := \langle \mathbb{R}, B_< \rangle$. That is, the class **SCDLBWE** of those elements **CDLBWE** that meet the separability condition has $\mathfrak{R}$ as its only—up-to-isomorphism—element.

A negative result about $\mathcal{H}_\beta(@, \mathsf{E})$

- Let $\mathfrak{I} := \langle [0,1], < \rangle$ be the standard binary frame on the closed interval $[0,1]$.
- Let us consider the product frame

$$\mathfrak{R} \times \mathfrak{I} := \langle \mathbb{R} \times [0,1], <_\ell \rangle$$

  where $<_\ell$ is the lexicographic order on $\mathbb{R} \times [0,1]$ obtained from $<$.
- As $\mathfrak{R} \times \mathfrak{I}$ is complete but not separable, the betweenness frame

$$(\mathfrak{R} \times \mathfrak{I})^* := \langle \mathfrak{R} \times \mathfrak{I}, B_{<_\ell} \rangle$$

  is an element of **CDLBWE**, which is not in **SCDLBWE**.
- We can use this fact to show that separability is not expressible in the modal language extended with either satisfaction operators or existential modalities.

Frame Title

Theorem

Separability is not expressible in $\mathcal{H}_\beta(@, \mathsf{E})$.

Proof.

Given a class of frames **K** let $L(\mathbf{K})$ be the logic of **K**, i.e., the set of all formulas that are valid on every frame in **K**. For a single frame $\mathfrak{F}$, let $L(\mathfrak{F}) := L(\{\mathfrak{F}\})$. We have shown that

$$L(\mathbf{CDLBWE}) = L(\mathfrak{R}) \, .$$

Since $(\mathfrak{R} \times \mathfrak{I})^* \in \mathbf{CDLBWE}$, it follows that $L(\mathfrak{R}) \subseteq L((\mathfrak{R} \times \mathfrak{I})^*)$. So, if there were a formula φ characterizing separability of $\mathfrak{R}$, it would have to be in $L(\mathfrak{R})$, and so in $L((\mathfrak{R} \times \mathfrak{I})^*)$, which is impossible. □

Corollary

The countable chain condition is not expressible in $\mathcal{H}_\beta(@, \mathsf{E})$, so the class of Suslin lines cannot be captured in the language.

Thank You!

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