Generalized compactness in mathematics

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Some consequences of compactness $L_{\kappa,\kappa}$

There are many natural concepts in mathematics formulated in terms of compactness:[1] given an infinite cardinal κ and a structure A of size κ, is it the case that a given property φ holds in A if and only if φ holds in all substructures of A of size $< \kappa$? Consider the following examples for a cardinal $\kappa \geq \omega_2$:

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[1] There are various notions of compactness, we will focus on compactness of $L_{\kappa,\kappa}$, a straightforward generalization of the first-order compactness.

1. Suppose P of size κ is a partially ordered set such that all suborders of size $< \kappa$ can be decomposed into countably many chains. Does it follow that P can be decomposed into countably many chains?
2. Suppose T is a tree of size κ and every subtree of size $< \kappa$ can be decomposed into countably many antichains. Does it follow that T can be decomposed into countably many antichains?
3. Suppose G is a graph of size κ and all its subgraphs of size $< \kappa$ have a countable chromatic number. Does it follow that G has a countable chromatic number?
4. Suppose A is an abelian group of size κ and all its subgroups of size $< \kappa$ are free. Does it follow that A is free?

Since κ is uncountable, the properties φ in these examples are not first-order and therefore are not entailed by compactness of the usual first-order logic (denoted $L_{\omega,\omega}$).

However, φ is often expressible in an infinitary logic $L_{\kappa,\kappa}$, which allows formulas of length $< \kappa$ with $< \kappa$ many quantifiers and connectives. If κ is compact for $L_{\kappa,\kappa}$ and theories of size κ—we call such κ *weakly compact*,[2]—then all four questions above are answered positively.

However, every weakly compact cardinal is necessarily inaccessible, hence quite far away from the usual objects in mathematics (in terms of size).

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[2] If there is no restriction on the size of theories, κ is called *strongly compact* and the examples (1)–(4) are true for structures of unlimited size with respect to substructures of size $< \kappa$.

One way of bringing the consequences of weak compactness down to small cardinals is to consider only specific principles which might consistently hold at accessible cardinals.

For instance, (2) can consistently hold at $\kappa = \omega_2$ (Rado's Conjecture, RC) and (4) can hold at $\kappa = \aleph_{\omega^2+1}$. However, for (1) (Galvin's Conjecture) and (3) it is still open whether they can hold below a weakly compact cardinal. These examples illustrate that it is unclear, a priori, which principles can consistently hold for small cardinals, and whether there are uniform methods to discover them.

We will survey recent development in this area, with a broader goal in mind of discussing whether compactness principles are good candidates for axioms in mathematics.

This goal is an updated version of the original program proposed by Gödel (1947) to look for consequences of large cardinal axioms in order to decide independent statements like the Continuum Hypothesis, CH.

By Solovay's observation that a large cardinal κ is preserved by all forcings of size $< \kappa$, Gödel's program necessarily fails for independent statements whose truth can be changed by small forcings. These include CH and many other, for instance Suslin Hypothesis, Whitehead's Conjecture (regarding the existence of non-free W groups) or Baumgartner's axiom (categoricity of ω_1-dense subsets of reals).

We will discuss whether compactness fares better in this respect.

Forcing axioms

One can broaden Gödel's program even further, and include *forcing axioms* as candidates for new axioms. Axioms like PFA, Proper Forcing Axiom, and MM, Martin's Maximum, were shown to be extremely powerful and capable of deciding almost all traditional independent problems in mathematics (which are typically decided by $V = L$ in the opposite way).

Even though forcing axioms and compactness principles are sometimes treated as two distinct concepts, they share structural similarity because they can both be formulated in terms of the existence of certain non-principal ultrafilters on infinite Boolean algebras.

- Compactness of $L_{\kappa,\kappa}$ generalizes the Boolean Prime Ideal Theorem, BPI, and asserts the existence of non-trivial ultrafilters on Boolean algebras which are closed under *countable* intersections.
- Forcing axioms generalize the Baire Category Theorem, BC, by requiring that for all Boolean algebras in a certain class, there are ultrafilters which meet any given list of ω_1-many dense open subsets.

The inherent limitation of forcing axioms to the cardinal $2^\omega = \omega_2$ suggests that some other principles—such as compactness—might be considered to decide properties of larger structures.

The investigation of compactness has the additional benefit of identifying principles which go beyond forcing axioms: some compactness principles are provably false at ω_2 (for instance compactness for abelian groups or chromatic compactness of graphs, or incompatible with forcing axioms (for instance Rado's Conjecture).

Indestructibility, or preservation

Some compactness principles originating from the compactness of $L_{\kappa,\kappa}$ are preserved by large classes of forcing notions, i.e., if they hold in V, they continue to hold in $V[\mathbb{P}]$, whenever $\mathbb{P}$ belongs to the said class.

- This might be interpreted positively from the philosophical perspective as lending a degree of robustness and stability to these principles.
- However, it also prevents them from deciding independent statements whose truth can be changed by forcings from these classes.

Some problems in mathematics independent from ZFC

- The Continuum Hypothesis
- Suslin Hypothesis, SH.[3]
- Whitehead's Conjecture, WC.[4]
- Baumgartner's Axiom, BA.[5]

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[3] Separability of $(\mathbb{R}, <)$ can be replaced by ccc. $\Diamond_{\omega_1}$ implies $\neg$SH (Jensen, 1972), MA_{ω_1} implies SH (Solovay–Tennenbaum, 1971).

[4] An abelian group G is *Whitehead* if every surjective homomorphism from any abelian group A onto G with kernel $\mathbb{Z}$ splits. Every free group is Whitehead. Whitehead asked whether the converse holds as well. Stein (1951) proved that all countable Whitehead groups are free. We write WC(κ) to assert that there exists a non-free Whitehead group of size κ (a counterexample to all Whitehead groups being free). By Shelah (1974), MA_{ω_1} implies WC(κ) for every regular uncountable κ, while $\Diamond_{\omega_1}(S)$ for every stationary S implies $\neg$WC(ω_1) (in $V = L$, $\neg$WC(κ) for all regular uncountable κ).

[5] All ω_1-dense subsets of $\mathbb{R}$ are oreder-isomorphic. CH implies the failure of BA (Sierpińsky 1950), while PFA proves BA (Baumgartner, 1984).

- Kaplansky's Conjecture, KP.[6]
- . . .

We will observe that none of these problems is decided by some of the strongest compactness principles at ω_2, in contrast with PFA. For KP, there are some open questions.

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[6] Every algebra homomorphism from $C(X)$, where X is any infinite compact Hausdorff space and $C(X)$ is the Banach algebra of continuous real valued functions, into any other commutative Banach algebra is continuous ("automatic continuity"). CH implies ¬KP and PFA implies KP (Esterle, Dales and Woodin, Todorcević, 1980's). It is still open whether ¬KP is consistent with $2^\omega > \aleph_\omega$. In particular, jumping ahead, it is open whether ¬KP is consistent with tree properties (but it is consistent with stationary reflection, in fact with FRP, and $2^\omega = \omega_2$) (we will skip this proof).

"Logical" compactness principles

Theorem (Jech 1973, Magidor 1974)

Suppose κ is a regular uncountable cardinal. Then:

1. *κ is strongly compact iff κ is inaccessible and* TP_κ *(equivalently* SP_κ*) holds.*
2. *κ is supercompact iff κ is inaccessible and* ITP_κ *(equivalently* ISP_κ *or* GMP_κ*, Viale, Weiss 2010-20s) holds.*

These principles are formulated using a two-dimensional generalization of trees: so called (κ, λ)-lists (which can be thin, or more generally slender) and cofinal or ineffable branches.

We will not give the definitions here, but we will mention some key consequences of these principles at small cardinals.

Consequences, with focus on ω_2

In what follows, a principle without a cardinal index is assumed to be formulated for ω_2.

- Tree properties at ω_2 imply $\neg$CH. ISP (in fact $\neg\mathsf{wKH}_{\omega_1}$) implies $2^\omega = 2^{\omega_1}$ if $2^\omega < \aleph_{\omega_1}$, Lambie-Hanson–Stejskalova (2024).
- ISP implies the failure of the approachability property at ω_2 and the negation of the weak Kurepa Hypothesis at ω_1 (Krueger 2017). ITP is not sufficient for these results.
- ISP implies SCH (Krueger 2019).

- ITP implies $\mathrm{AD}^{L(\mathbb{R})}$, and hence Projective Determinacy. Weiss 2012 proved it implies $\neg\square(E^{\lambda}_{<\omega_2}, \omega_2)$, where $E^{\lambda}_{<\omega_2}$ is the set of ordinals below λ of cofinality $< \omega_2$, for all λ with $\mathrm{cf}(\lambda) \geq \omega_2$.[7] By Steel's theorem 2005 the failure of a square at a strong limit singular cardinal implies $\mathrm{AD}^{L(\mathbb{R})}$, hence ITP_κ for any $\kappa \geq \omega_2$ entails $\mathrm{AD}^{L(\mathbb{R})}$.

All these consequences thus follow already from the "compactness" part of PFA.

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[7] In $\square_\lambda(E, \kappa)$, the first parameter in the brackets is the domain of the principle, the second one is the width and the subscript is an (optional) order-type restriction (the sequences are required to have length $\leq \lambda$).

Preservation

- Honzik, Lambie-Hanson and Stejskalova showed (2024) that Cohen forcing at a regular cardinal μ preserves $\mathsf{ISP}_{\mu^{++}}$ over all models where this principle holds.
  Note preservation by single Cohen at μ for TP and ITP is open.
- Lambie-Hanson and Stejskalova recently observed that (2024) also establishes that Cohen forcing at a regular cardinal μ preserves $\neg\mathsf{wKH}(\mu^+)$ over all models where it holds.
  Note that the preservation by single a Cohen real at μ of KH_{ω_1} is open.

We will observe that this preservation implies easily that Whitehead's Conjecture and Suslin's Hypothesis are independent from ISP.

“Mathematical” compactness principles

Suppose $V = L$. Then the following are equivalent to weak compactness of κ:

Theorem

Logic and trees.

- *κ is weakly compact for the infinitary logic $L_{\kappa,\kappa}$ with signature of size κ.*
- *There are no κ-Suslin trees, and no κ-Aronszajn trees.*

Squares and stationary reflection.

- *All stationary subsets of $E^\kappa_\omega = \{\alpha < \kappa \,|\, \mathrm{cf}(\alpha) = \omega\}$ reflect.*
- *Stationary reflection $\mathrm{SR}(\kappa)$ holds.*
- *Todorčević’s square $\square(\kappa)$ does not hold.*
- *Fodor-type Reflection principle $\mathrm{FRP}(\kappa)$ holds.*
- *κ is $\Delta_{<\kappa,\kappa}$-compact for κ-sized algebras.*

Theorem

Algebras.

- κ *is abelian compact for* κ*-sized abelian groups.*

Graphs.

- κ *is countably coloring compact for* κ*-sized graphs.*
- κ *is countably chromatically compact for* κ*-sized graphs.*
- *Rado's Conjecture for graphs of size* κ *holds.*

Topological spaces.

- κ *is collectionwise Hausdorff compact for* κ*-sized topological spaces.*

Stationary reflection

Definition

If κ is regular and $S \subseteq \kappa$, we say that S *reflects* if there is $\alpha < \kappa$ of uncountable cofinality such that $S \cap \alpha$ is a stationary subset of α. We say that $S \subseteq \kappa$ is *non-reflecting* if S does not reflect. We write SR(κ) if all relevant stationary subsets of κ reflect.

Stationary reflection is necessary for compactness of mathematical structures we mentioned above (in contrast to logical principles which are independent from it), but it is known that it is not strong enough to be a sufficient condition as well. Several concepts have been introduced which provide non-trivial strengthenings of stationary reflection with compactness-type consequences: *Reflection Principle*, *Fodor-type Reflection Principle* and Δ*-reflection*.

Countable coloring and chromatic compactness

Suppose $\mathcal{G} = (G, E)$ is an (undirected) graph and $<$ some fixed wellorder on G. The neighborhood $N^{<}_{\mathcal{G}}(x)$ of a vertex $x \in G$ with respect to $<$ is defined by $N^{<}_{\mathcal{G}}(x) = \{y \mid \{x, y\} \in E \text{ and } y < x\}$.

Definition

The *coloring number* number of $\mathcal{G}$, $\mathrm{Col}(\mathcal{G}) = \chi$, is defined to be the least cardinal χ such that there is a wellorder $<$ on G such that $|N^{<}_{\mathcal{G}}(x)| < \chi$ for all $x \in G$.

The notion of the coloring number can be seen as a more constructive version of the usual chromatic number of a graph $\mathcal{G} = (G, E)$: a function $c : G \to \chi$ is called a chromatic coloring of $\mathcal{G}$ if $\{x, y\} \in E$ implies $c(x) \neq c(y)$ for all $x, y \in G$.

Definition

The *chromatic number* of $\mathcal{G}$, $\mathrm{Chr}(\mathcal{G})$, is defined as the least cardinal χ such that there is a chromatic coloring with range χ.

The wellorder $<$ in the definition of colorwise compactness provides an explicit construction of a chromatic function with small domain, hence $\mathrm{Chr}(\mathcal{G}) \leq \mathrm{Col}(\mathcal{G})$.

Definition

We say that $\kappa \geq \omega_2$ is *countably color-compact or chromatically-compact* if for any graph of size κ if all its strictly smaller subgraphs have a countable coloring or chromatic number, respectively, the whole graph is countably compact.

FRP

Definition

Let κ be a regular cardinal $\geq \omega_2$. The *Fodor-type Reflection Principle* for κ, FRP(κ), is the following statement: For any stationary $S \subseteq E^\kappa_\omega = \{\alpha < \kappa \,|\, \mathrm{cf}(\alpha) = \omega\}$ and a mapping $g : S \to [\kappa]^{\leq\omega}$ there is $I \in [\kappa]^{\omega_1}$ such that

1. $\mathrm{cf(I)} = \omega_1$,
2. $g(\alpha) \subseteq I$ for all $\alpha \in I \cap S$,
3. For any regressive $f : S \cap I \to \kappa$ such that $f(\alpha) \in g(\alpha)$ for all $\alpha \in S \cap I$, there is $\xi^* < \kappa$ such that $f^{-1}{}''\{\xi^*\}$ is stationary in $\sup(I)$.

Δ-reflection

Definition (Magidor–Shelah (1994))

Suppose $\lambda < \kappa$ are uncountable cardinals, with κ regular. $\Delta_{\lambda,\kappa}$ is the statement that, for every stationary $S \subseteq \{\alpha < \kappa \,|\, \mathrm{cf}(\alpha) < \lambda\}$ and every algebra A on κ with fewer than λ operations, there is a subalgebra A^* of A such that, letting $\delta = \mathrm{otp}(A^*)$, the following hold:

1. δ is a regular cardinal;
2. $\delta < \lambda$;
3. $S \cap A^*$ is stationary in $\sup(A^*)$.

We say that κ has the $\Delta_{<\kappa,\kappa}$-reflection if $\Delta_{\lambda,\kappa}$ holds for every $\lambda < \kappa$. We say that κ has the global Δ_κ-reflection if $\Delta_{\kappa,\nu}$ holds for all regular $\nu > \kappa$.

Down ω_2

- FRP can be brought down to ω_2 so that FRP(κ) holds for all regular $\kappa \geq \omega_2$. It is equivalent to various mathematical principles:
  - By Fuchino and Rinot (2011), FRP is equivalent to the statement that every Boolean algebra is openly generated if and only if it is ω_2-projective.
  - By Fuchino et al. (2019) , FRP is equivalent to countable color compactness of all graphs, to collectionwise Hausdorff compactness, and to "For any locally countable compact topological space X, if all subspaces of X of cardinality $\leq \omega_1$ are metrizable, then X is also metrizable".

  FRP is equivalent to several compactness properties for specific mathematical structures:

Down to $\aleph_{\omega^2+1}$

By results of Magidor–Shelah (1994):

- Δ-reflection can be brought down to $\aleph_{\omega^2+1}$ (optimal for structures of size $\aleph_{\omega^2+1}$).
- Size-unrestricted Δ-reflection can be brought down to the first cardinal fixed point of the $\aleph$ function $\kappa = \aleph_\kappa$.

Δ-reflection implies countable color compactness and abelian compactness. Hence it is consistent, for example, that ZFC cannot prove (modulo large cardinals) the existence of almost-free non-free abelian groups larger than the first cardinal fixed point. However, there are always almost-free non-free abelian groups below $\aleph_{\omega^2}$. In particular, abelian compactness does not follow from forcing axioms.

Down to ω_2, for weaker principles

Countable chromatic compactness cannot be brought below $\beth_\omega$ (weak compactness is the best lower bound known, however). However, compactness for smaller classes of graphs can be brought down.

Rado's Conjecture

- Galvin's Conjecture; in the language of graphs related to chromatic numbers of incomparability graphs.[8] It is consistent that for any partially ordered set P, P can be decomposed into countably many chains if and only if every suborder P' of size $\leq \omega_1$ can be decomposed into countably many chains.
- Rado's Conjecture, RC; in the language of graphs related to chromatic numbers of interval graphs.[9] It is consistent that for any tree T, T can be decomposed into countably many antichains if and only if every subtree T' of size $\leq \omega_1$ can be decomposed into countably many antichains.

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[8] Graphs of the form (G, E) where G is the domain of a partially ordered set $(P, <)$ (or more generally a quasi-ordered set) and $\{x, y\} \in E$ iff x, y are incomparable in $<$.

[9] Graphs of the form (G, E) where G is the set of intervals (or more generally convex sets) of some linearly ordered set $(L, <)$ and $\{I, J\} \in E$ iff $I \cap J \neq \emptyset$.

Galvin's Conjecture is still open. Todorčević showed (1983) that RC is relatively consistent with the existence of a strongly compact cardinal: if a strongly compact cardinal κ is turned into ω_2 using Levy collapse, then $\mathsf{RC} + \mathsf{CH}$ holds in the resulting model. Zhang (2020) elaborated on Todorčević's observation that RC holds in the Mitchell model and showed that $\mathsf{RC} + \mathsf{TP}_{\omega_2}$ is consistent from a strongly compact cardinal.

RC contradicts MA_{ω_1}, which makes it rather exceptional because it provides an alternative to MM. It implies, among other things, $2^\omega \leq \omega_2$, the failure of $\square(\kappa)$ for any regular $\kappa \geq \omega_2$, and hence $\mathsf{AD}^{L(\mathbb{R})}$, and also the Strong Chang's Conjecture and Weak Reflection Principle at ω_2. We will mention the theories $\mathsf{ZFC} + \mathsf{MM}$ and $\mathsf{ZFC} + \mathsf{RC} + 2^\omega = \omega_2$ as two examples of unification of logical and mathematical principles.

Separations: is compactness weak for traditional problems?

Theorem (H. 2025)

The principles SH, WC *and* BA *are independent from* ZFC + ISP + FRP. *Moreover,* ¬BA, ¬WC *and* ¬SH *are consistent with* $\mathrm{ZFC} + \mathrm{RC} + 2^{\omega} = \omega_2$.

In this notation $V = L$ implies the negative versions of these principles. WC denotes the Whitehead's Conjecture for abelian groups of size ω_1.

The first part of the theorem follows because both ISP and FRP are preserved by the Cohen forcing at ω. RC is destroyed by adding a single real, so a direct argument is applied.

We will sketch the proof that $\neg$WC is consistent with $\mathrm{ZFC} + \mathrm{RC} + 2^{\omega} = \omega_2$.

We start by stating a theorem Bergfalk et al. (2024) that any number of Cohen reals of cofinality at least ω_2 falsifies WC:

Theorem

Suppose A is a non-free abelian group of size ω_1. Then in $V[\mathrm{Add}(\omega, \omega_1)]$, A is not Whitehead. Moreover, if $\mathbb{P}$ is a ccc forcing in V, then A stays non-Whitehead in $V[\mathrm{Add}(\omega, \omega_1) \times \mathbb{P}]$.

Suppose A is a non-free abelian group of size ω_1 in $V[\mathbb{M}]$, where $\mathbb{M}$ is a sparse version of the Mitchell forcing (collapses appear at cofinalities $> \omega_1$). We wish to show that A is non-Whitehead (we will write "non-W") in $V[\mathbb{M}]$. By Zhang 2020, this models satisfies RC.

Due to $\mathbb{M}$ being κ-cc, the group A appears at some stage $V[\mathbb{M}_\alpha]$, $\alpha < \kappa$. Let us work in $V[\mathbb{M}_\alpha]$. A is non-free in $V[\mathbb{M}_\alpha]$ due to the downward preservation of this property. By Bergfalk et al. the tail of the Cohen forcing $\mathrm{Add}_{[\alpha,\kappa)}$ makes A non-W. It follows that A is a non-W group in:

$$V[\mathbb{M}_\alpha][\mathrm{Add}_{[\alpha,\kappa)}].$$

Let us fix a homomorphism $f : B \to A$ in $V[\mathbb{M}_\alpha][\mathrm{Add}_{[\alpha,\kappa)}]$ which does not split. Both f and B have size ω_1, so we can assume by permuting the generic for $\mathrm{Add}_{[\alpha,\kappa)}$ if necessary that f, B are added by $\mathrm{Add}_{[\alpha,\alpha+\omega_1)}$ over $V[\mathbb{M}_\alpha]$. Let us denote $\alpha + \omega_1$ by β.

Since $\mathbb{M}$ is sparse, there are no collapses in the interval $[\alpha, \beta)$, hence also the model $V[\mathbb{M}_\beta]$ contains $f : B \to A$ which does not split (in $V[\mathbb{M}_\beta]$). Let us work over $V[\mathbb{M}_\beta]$, and let $\mathbb{P}$ denote the tail of the Cohen forcing and $\mathbb{T}$ the ω_1-closed term forcing, with $\mathbb{T} \times \mathbb{P}$ projecting onto the tail of the Mitchell forcing. We will argue that $\mathbb{T} \times \mathbb{P}$ does not add a splitting homomorphism to f. It follows that there cannot be a splitting homomorphism in $V[\mathbb{M}]$, which finishes the proof.

This is shown using the standard method of working in $V[\mathbb{M}_\beta]$ and diagonalizing over antichains in $\mathbb{P}$ and building a decreasing sequence of conditions in $\mathbb{T}$ by recursion on ω_1.

In some detail, suppose for contradiction that $\dot{g}$ is forced by $\mathbb{P} \times \mathbb{T}$ over $V[\mathbb{M}_\beta]$ to be a splitting homomorphism for $f : B \to A$ which is not in $V[\mathbb{M}_\beta][\mathbb{P}]$, and let $\langle a_\alpha \,|\, \alpha < \omega_1 \rangle$ be some enumeration of A. Build a decreasing sequence in $\mathbb{T}$, $\langle t_\alpha \,|\, \alpha < \omega_1 \rangle$, and a sequence of antichains $\langle X_\alpha \,|\, \alpha < \omega_1 \rangle$ in $\mathbb{P}$ such that if G is any generic for $\mathbb{P}$, then in $V[\mathbb{M}_\beta][G]$, there is for each X_α exactly one condition $p_\alpha \in X_\alpha \cap G$ and (p_α, t_α) determines the value of $\dot{g}(a_\alpha) := b_\alpha$ in B. It is easy to see that the function in $V[\mathbb{M}_\beta][G]$ which maps a_α to b_α is a splitting homomorphism in $V[\mathbb{M}_\beta][G]$, which is a contradiction.

Unifications

A natural principle unifying the logical and many mathematical principles is MM: it implies FRP (and hence all its mathematical consequences we mentioned) and ISP. However, some points can be considered:

- There are other unifying principles for ω_2 which are incompatible with forcing axioms, such as Rado's Conjecture, itself a consequence of the *Game Reflection Principle*, $\mathrm{GRP}^{<\omega_1}(<\omega_2)$, introduced by König 2004, which is equivalent to generic supercompactness of ω_2 for σ-closed forcings.[10] However, from the perspective of unifying logical and mathematical compactness principles, $\mathrm{GRP}^{<\omega_1}(<\omega_2)$ is less appealing because it implies CH by Fuchino (2021).

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[10] See Fuchino (preprint) for a comprehensive account of Laver-generic large cardinals, which are related to this topic. See also Foreman 1998 for a clear summary of motivations and consequences of generic compactness, with some comments on the potential of these principles for becoming new axioms of set theory.

- Mathematical principles for cardinals above ω_2, dealing with uncountable compactness—that is, with stationary subsets of $[\kappa]^\theta$ for an uncountable θ—are much less understood, largely because of provable differences between $[\kappa]^\omega$ and $[\kappa]^\theta$, $\theta > \omega$, which make the latter structure less tractable. The lack of uniformity suggests that the unifications found at ω_2 are an exception, not a rule, thereby positioning MM as an isolated principle, rather than an instance of a more general structure.

Consider the following theory:

$$T^{+} :=_{\mathsf{df}} \mathsf{ZFC} + \mathsf{RC} + 2^{\omega} = \omega_2,$$

which may be a genuine and powerful alternative to ZFC + MM. By results of Torres-Pérez and Wu (2017), the strong form of Chang's conjecture with ¬CH implies the strong tree property at ω_2, TP_{ω_2} in our notation. Thus T^{+} proves TP_{ω_2}, unifying some logical and mathematical principles.[11]

RC is fragile, just as MM is, in terms of indestructibility under forcing, but as we mentioned above using several examples, it does not decide the traditional problems the way MM does (it is open whether they are independent from RC or not).

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[11] However, it is known that T^{+} does not prove ITP_{ω_2} by Zhang 2020. The reason is that RC is consistent just from strong compactness and thus principles related to supercompactness appear to be outside its reach.

Although T^+ has potential for generalization in terms of cardinalities (unlike MM), the main research focus remains on ω_2, where the most interesting applications and problems are found. Nonetheless, proving interesting statements from a generalized version of T^+ would strengthen its standing as a specific instance of a global pattern.

More details can be found in an ariticle "Compactness for small cardinals in mathematics: principles, consequences, and limitations" (R. Honzik) in ArXiv.