On Fuzzy Logic, Probability Theory, and Their Cooperation

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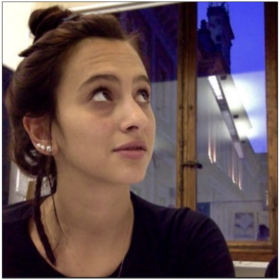
Results are in collaboration with:



Sara Ugolini (IIIA - CSIC, Spain) - **Sandro Preto** (ABC-Univ. Brazil)

long discussions with Lluis Godo, Franco Montagna, Daniele Mundici, Esther Corsi, Hykel Hosni, Mirko Navara, Daniil Kozhemiachenko, Stefano Aguzzoli, Vincenzo Marra, Serafina Lapenta, Luca Spada, Antonio Di Nola, Barbara Vantaggi

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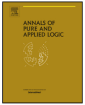

Encoding de Finetti's coherence within Łukasiewicz logic and MV-algebras

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Main Track

Reasoning About Probability via Continuous Functions

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The main inspirations for this talk comes from **Zadeh**'s paper

Probability Theory and Fuzzy Logic Are Complementary Rather Than Competitive[1]

and the paper of **Hájek**, **Godo** and **Esteva**

Fuzzy logic and Probability[2]

We would like to offer a further angle to regarded these two theories and to discuss the following claim:

While **complementary** in applications, they act in **cooperation** in their logico-mathematical formalization.

[1] Zadeh. Discussion: Probability Theory and Fuzzy Logic Are Complementary Rather Than Competitive *Technometrics, 1995.*

[2] Hájek, Godo, Esteva. Fuzzy logic and Probability *Proc. of Uncertainty in Artificial Intelligence, 1995.*

The Ground Ideas

Introduction

Probability theory: is a branch of logic (philosophy, mathematics, computer science) that studies quantification of the **uncertainty** of unknown events and their combinations.

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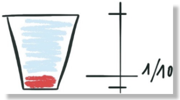
Fuzzy logic: is a branch of logic (philosophy, mathematics, computer science) that studies quantification of the **vagueness** (or imprecision) of known events and their combinations.

INTRODUCTION



A drink that is poisonous with truth-degree 0.1

Vs

A drink with probability 1/10 to be poisonous

Introduction

Besides the obvious philosophical gaps between them, probability theory and fuzzy logic **share many important properties** from the perspective of their mathematical formalization:

- formulas (events) are mapped into a totally ordered scale by the models of both theories, usually the **real unit interval** $[0, 1]$ (but other options are possible)
- classical logic (i.e., $\{0, 1\}$-valued models) is a particular case of both theories: it is the logic of **certainty** (particular case of uncertainty) and of **precision** (particular case of vagueness).

but they also have key formal distinctions:

- **Truth-functionality**: probability theory is **not** while fuzzy logic **is** (in Hájek's framework)
- **Algebrizability** (more precisely **structurality**): probability theory is **not** while fuzzy logic **is** (in Hájek's framework, see Carles Noguera's thesis for instance)

INTRODUCTION

Algebraizability for a logic is a useful tool for its **bridge theorems** that allow to study algebraically the logical problems of

- **Interpolation**: if $\gamma \vdash \varphi$ then one can find ψ on the same variables of γ and ψ such that
$$\gamma \vdash \psi \text{ and } \psi \vdash \gamma;$$
- **Unification**: given formulas $\gamma_1, \ldots, \gamma_n$ find (if it exists) a **unifier** for them, i.e. a substitution σ s.t.
$$\sigma(\gamma_1) = \sigma(\gamma_2) = \ldots = \sigma(\gamma_n) \text{ (up to an equational theory).}$$
- **Generalization**: given formulas $\gamma_1, \ldots, \gamma_n$ find (if it exists) a **generalization** for them, i.e. a formula ψ and substitutions $\sigma_1, \ldots, \sigma_n$ s.t. for all i,
$$\sigma_i(\psi) = \gamma_i \text{ (up to an equational theory).}$$

and other properties like the **functional representation** of formulas.

The Logic FP(Ł)

Łukasiewicz logic

The ground logic we use is Łukasiewicz logic (Ł)

- A language made of countably many variables $x_1, x_2, \ldots$
- Connectives $(\oplus, \neg, \bot)$ of type $(2, 1, 0)$
- Formulas are determined as usual and further connectives are definable:

$$\varphi \to \psi := \neg\varphi \oplus \psi;\ \varphi \vee \psi := (\varphi \to \psi) \to \psi;\ \varphi \wedge \psi := \neg(\neg\varphi \vee \neg\psi);$$
$$\varphi \ominus \psi := \neg(\varphi \to \psi);\ \top := \neg\bot$$

Łukasiewicz logic is **algebraizable** and its equivalent algebraic semantics is the variety $\mathbb{MV}$ of **MV-algebras**, generated by

$$[0,1]_{\mathbb{MV}} = \langle [0,1], \oplus, \neg, 0 \rangle$$

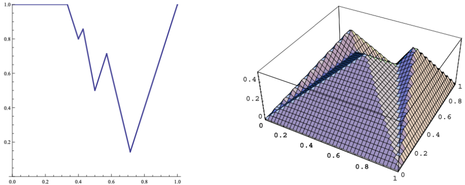
where for all $a, b \in [0,1]$, $a \oplus b = \min\{1, a+b\}$ and $\neg a = 1 - a$.

The MV-algebra $[0,1]_{\mathbb{MV}}$ plays for Łukasiewicz logic the same role that $\{0,1\}_{\mathbb{B}}$ plays for classical logic.

Łukasiewicz logic

McNaughton Theorem

The Lindenbaum-Tarski algebra $\mathbf{F}^k_{\mathbb{MV}}$ of Łukasiewicz logic over k variables is isomorphic to the algebra of functions $f : [0,1]^k \to [0,1]$ that are **continuous** and **piecewise linear** with **integer coefficients** with operations $\oplus$ and $\neg$ as in $[0,1]_{\mathbb{MV}}$, pointwisely.



(a) $f_1 \in \boldsymbol{F}^1_{\mathrm{MV}}$. (b) $f_2 \in \boldsymbol{F}^2_{\mathrm{MV}}$.

(Aguzzoli, Bova, Gerla. Free Algebras and Functional Representation for Fuzzy Logics. Chapter IX. handbook of MFL, 2011.)

The language of FP(Ł)

We expand the language of Łukasiewicz logic by a unary connective P and, for a classical formula φ, we read

$P(\varphi)$ as φ **is probable**.

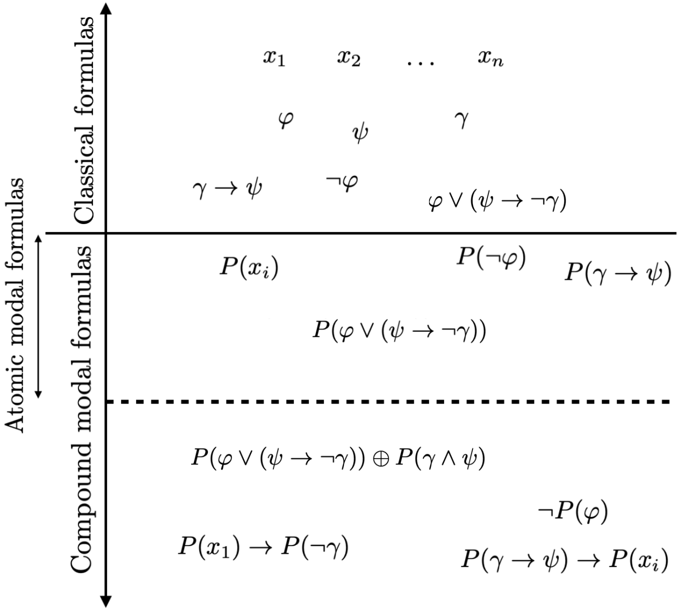
The formulas of FP(Ł) are Łukasiewicz formulas Φ where variables are basic formulas like $P(\varphi)$

$$\Phi[P(\varphi_1), P(\varphi_2), P(\varphi_3)] := P(\varphi_1) \oplus (P(\varphi_2) \to (P(\varphi_1) \ominus P(\varphi_3)))$$

We hence distinguish:

- A **first layer** made of *events*, i.e., classical formulas;
- A **second layer** made of *modal formulas*, i.e., atomic and compound probability formulas.

Notice that P acts as a partial operator: neither $\varphi \to P\psi$, nor $P(P\varphi \to P\psi)$ are wffs.


Classical formulas
Atomic modal formulas
Compound modal formulas
x_1 x_2 $\dots$ x_n
φ ψ γ
$\gamma \to \psi$ $\neg\varphi$ $\varphi \vee (\psi \to \neg\gamma)$
$P(x_i)$ $P(\neg\varphi)$ $P(\gamma \to \psi)$
$P(\varphi \vee (\psi \to \neg\gamma))$
$P(\varphi \vee (\psi \to \neg\gamma)) \oplus P(\gamma \wedge \psi)$
$\neg P(\varphi)$
$P(x_1) \to P(\neg\gamma)$
$P(\gamma \to \psi) \to P(x_i)$

THE AXIOMS

Classical formulas obey to axioms and rules of **classical logic**.

Compound probability formulas obey to axioms and rules of **Łukasiewicz logic**.

Axioms and Rules for P, [Hájek, Esteva, Godo - '95]

(P1) $\neg P\bot$;

(P2) $P(\varphi \to \psi) \to (P\varphi \to P\psi)$;

(P3) $P(\varphi \vee \psi) \leftrightarrow ((P\varphi \to P(\varphi \wedge \psi)) \to P\psi)$.

(N) From φ, derive $P\varphi$.

Remark: By a result of Baldi, Cintula, Noguera[3] The logic FP(Ł) (a minimal extension of it) is essentially the same as the probability logic of Fagin, Halpern and Megiddo.

[3] Baldi, Cintula, Noguera. Classical and Fuzzy Two-Layered Modal Logics for Uncertainty: Translations and Proof-Theory. Int J Comput Intell Syst, 2020.

THE PROBABILITY MODELS

Standard models for FP(Ł) are **probability functions** $\mu : \mathbf{F}^n_{\mathbb{B}} \to [0,1]$.

For an atomic formula $P(\varphi)$,

$$\|P(\varphi)\|_\mu = \mu(\varphi).$$

For a compound formula $\Phi = \Phi[P(\varphi_1), \ldots, P(\varphi_k)]$,

$$\|\Phi\|_\mu = \Phi^{[0,1]_{\mathbb{MV}}}(\|P(\varphi_1)\|_\mu, \ldots \|P(\varphi_k)\|_\mu) = \Phi^{[0,1]_{\mathbb{MV}}}(\mu(\varphi_1), \ldots, \mu(\varphi_k)).$$

For formulas Φ and Ψ, we write $\Phi \models_\mu \Psi$ if for all μ such that $\|\Phi\|_\mu = 1$, then $\|\Psi\|_\mu = 1$ as well.

Theorem [Hájek, Godo, Esteva - '95]

If Φ and Ψ are formulas from FP(Ł), then $\Phi \vdash_{FP} \Psi$ iff for all probability μ such that $\models_\mu \Phi$, then $\models_\mu \Psi$ (and we write $\Phi \models_\mu \Psi$).

The Geometric Models

The Geometric Models

In 1931 Bruno de Finetti provided a general justification for the probabilistic representation of rational degrees of belief [4].

The **probability** of an unknown event *a* is the **fair price** that a rational Gambler is willing to pay to participate in a betting game, against the Bookmaker.

The payoff of the game is **1** in case *a* occurs, and **0** otherwise.

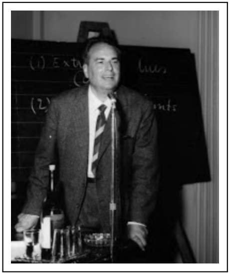
[4] De Finetti. Sul significato soggettivo della probabilità. Fundamenta Mathematicae, 1931.

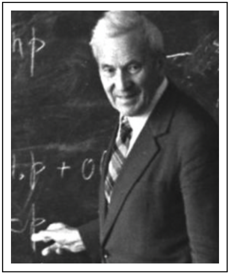
The Geometric Models

Theorem [de Finetti - '31]

Let $\mathcal{E} = \{\varphi_1, \ldots, \varphi_k\}$ be a finite set of events in, say, n variables and let $\beta : \mathcal{E} \to [0, 1]$ be any map. Then, β is **coherent** iff there exists a **probability function** μ that extends β over the φ_i's:

$$\beta(\varphi_i) = \mu(\varphi_i).$$





The Geometric Models

Every finite set $\mathcal{E} = \{\varphi_1, \ldots, \varphi_k\}$ of events let us consider the following set

$$\mathscr{C}_{\mathcal{E}} = \text{co}\{\langle \varphi_1(x), \ldots, \varphi_k(x)\rangle \mid x \in \{0,1\}^n\}$$

or, equivalently,

$$\mathscr{C}_{\mathcal{E}} = \text{co}\{\langle h(\varphi_1), \ldots, h(\varphi_k)\rangle \mid h \in \mathcal{H}(\mathbf{F}^n_{\mathbb{B}}, \{0,1\})\}$$

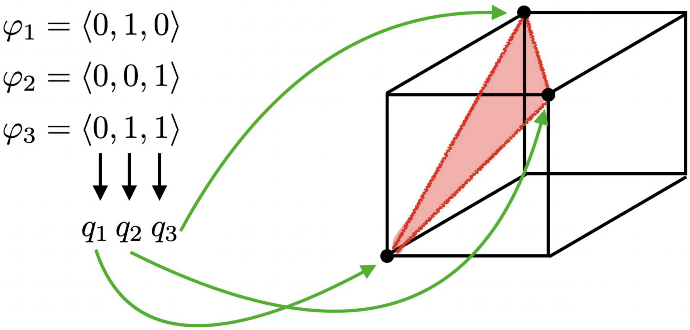
Proposition (Paris - ISIPTA01 & F., Ugolini - APAL24)

Let $\mathcal{E} = \{\varphi_1, \ldots, \varphi_k\}$ be any finite set of events and let $\beta : \mathcal{E} \to [0,1]$. Then,

$$\beta \text{ is coherent iff } \vec{\beta} = \langle \beta(\varphi_1), \ldots, \beta(\varphi_k)\rangle \in \mathscr{C}_{\mathcal{E}}.$$

Furthermore $\mathscr{C}_{\mathcal{E}}$ is a **rational polytope**, i.e., a finitely generated convex polyhedron with rational vertices.

The Geometric Models



The Geometric Models

Corollary (F., Ugolini - APAL24)

If Φ and Ψ are formulas from FP(Ł), on events in $\mathcal{E} = \{\varphi_1, \ldots, \varphi_k\}$ then the following are equivalent:

- $\Phi \vdash_{FP} \Psi$;
- for every probability μ, $\Phi \models_\mu \Psi$;
- for every coherent map $\beta : \mathcal{E} \to [0,1]$, $\Phi \models_\beta \Psi$;
- for every $\vec{\beta} \in \mathscr{C}_{\mathcal{E}}$, $\Phi \models_{\vec{\beta}} \Psi$.

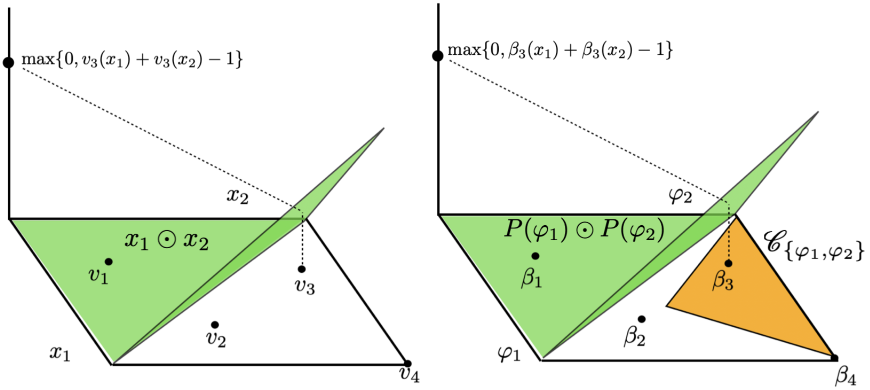
A COMPARISON

Łukasiewicz	FP(Ł)
$\varphi \vdash_{Ł} \psi$	$\Phi \vdash_{FP} \Psi$
for all $h : \mathbf{F}^k_{\mathbb{MV}} \to [0,1]_{\mathbb{MV}}$ $\varphi \models_h \psi$	for all probability μ, $\Phi \models_\mu \Psi$
for all $v : \{x_1, \ldots, x_k\} \to [0,1]$, $\varphi \models_v \psi$	for all coherent $\beta : \mathcal{E} \to [0,1]$, $\Phi \models_\beta \Psi$
for all $\mathbf{x} \in [0,1]^k$, $\varphi \models_x \psi$	for all $\vec{\beta} \in \mathscr{C}_{\mathcal{E}}$, $\Phi \models_{\vec{\beta}} \Psi$

A COMPARISON

Łukasiewicz	FP(Ł)
$\varphi \vdash_{Ł} \psi$	$\Phi \vdash_{FP} \Psi$
for all $h : \mathbf{F}^k_{\mathbb{MV}} \to [0,1]_{\mathbb{MV}}$ $\varphi \models_h \psi$	for all probability μ, $\Phi \models_\mu \Psi$
for all $v : \{x_1, \ldots, x_k\} \to [0,1]$, $\varphi \models_v \psi$	for all coherent $\beta : \mathcal{E} \to [0,1]$, $\Phi \models_\beta \Psi$
for all $\mathbf{x} \in [0,1]^k$, $\varphi \models_x \psi$	for all $\vec{\beta} \in \mathscr{C}_{\mathcal{E}}$, $\Phi \models_{\vec{\beta}} \Psi$

A COMPARISON



The Algebraic Models

Coherent MV-algebras

For a subset $\mathscr{S}$ of $[0,1]^k$, let us denote by $\mathbf{F}_{MV}(k)/\mathscr{S}$ the algebra of functions from $\mathbf{F}_{MV}(k)$ **restricted** to $\mathscr{S}$. By results and **Vincenzo Marra** and **Luca Spada**:

Semisimple MV-algebras are all of the form $\mathbf{F}^k_{\mathbb{MV}}/\mathscr{S}$ where $\mathscr{S}$ is a **closed** subset of $[0,1]^k$.[a]

Finitely presented MV-algebras are all of the form $\mathbf{F}^k_{\mathbb{MV}}/\mathscr{S}$ where $\mathscr{S}$ is a **polyhedron** of $[0,1]^k$.[b]

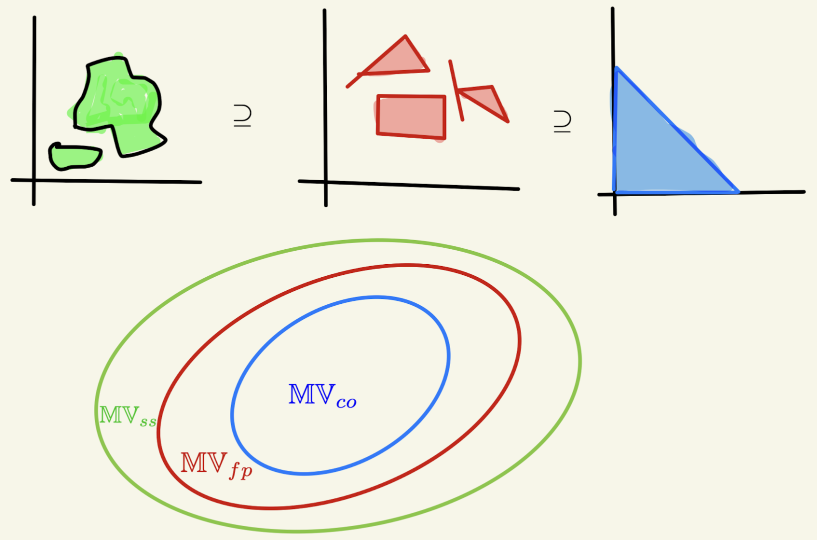
[a] Marra, Spada. The dual adjunction between MV-algebras and Tychonoff spaces, SL. 2012.

[b] Marra, Spada. Duality, projectivity, and unification in Łukasiewicz logic and MV-algebras, APAL. 2013.

Definition (F., Ugolini - APAL24)

Coherent MV-algebras are all of the form $\mathbf{F}^k_{\mathbb{MV}}/\mathscr{C}$ where $\mathscr{C}$ is a **coherent set** of $[0,1]^k$.

Coherent MV-algebras



Coherent MV-algebras

Take a formula of FP(Ł), $\Phi = \Phi[P(\varphi_1), \ldots, P(\varphi_k)]$, on events $\mathcal{E} = \{\varphi_1, \ldots, \varphi_k\}$.

1. Take the coherent set $\mathscr{C}_{\mathcal{E}}$ detemined by Φ.
2. Define the coherent MV-algebra $\mathbf{C} = \mathbf{F}^k_{\mathbb{MV}}/\mathscr{C}_{\mathcal{E}}$.
3. Let us map: or each $i = 1, \ldots, k$,

$$v : P(\varphi_i) \mapsto [\varphi_i]_{\mathscr{C}_{\mathcal{E}}} \in \mathbf{C},$$

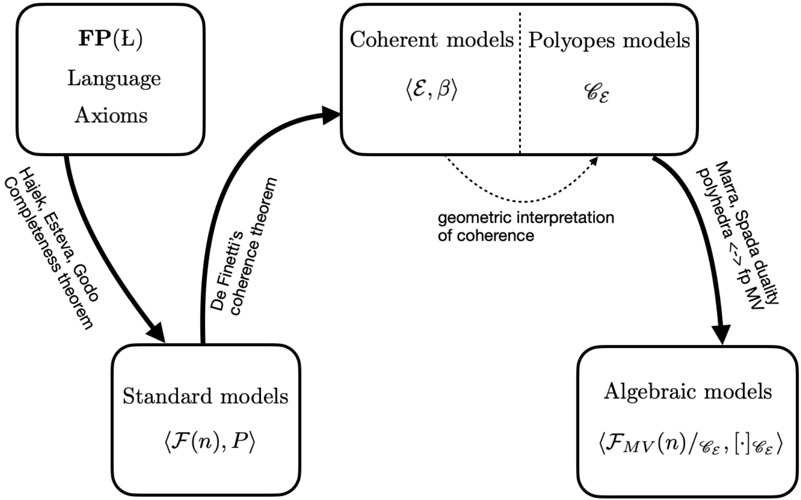
4. We write $\models^l_{\mathbf{C}} \Phi$ if $v(\Phi) = \top$ (in $\mathbf{C}$).

Coherent MV-algebras

Corollary (F., Ugolini - APAL'24)

If Φ and Ψ are formulas from FP(Ł), on events in $\mathcal{E} = \{\varphi_1, \ldots, \varphi_k\}$ then the following are equivalent:

- $\Phi \vdash_{FP} \Psi$;
- for every standard model $\langle \mathbf{F}(n), \mu \rangle$, $\Phi \models_\mu \Psi$;
- for all coherent maps $\beta : \mathcal{E} \to [0,1]$, $\Phi \models_\beta \Psi$;
- for all $\beta \in \mathscr{C}_\mathcal{E}$, $\Phi \models_\beta \Psi$;
- for the coherent MV-algebra $\mathbf{C} = \mathbf{F}^k_{\mathbb{MV}}/\mathscr{C}_\mathcal{E}$, $\Phi \models^l_{\mathbf{C}} \Psi$.


$\mathbf{FP}(Ł)$
Language
Axioms
Coherent models
$\langle \mathcal{E}, \beta \rangle$
Polyopes models
$\mathscr{C}_{\mathcal{E}}$
Hajek, Esteva, Godo
Completeness theorem
De Finetti's
coherence theorem
geometric interpretation
of coherence
Marra, Spada duality
polyhedra <-> fp MV
Standard models
$\langle \mathcal{F}(n), P \rangle$
Algebraic models
$\langle \mathcal{F}_{MV}(n)/_{\mathscr{C}_{\mathcal{E}}}, [\cdot]_{\mathscr{C}_{\mathcal{E}}} \rangle$

Applications

Prob. Unification

Given FP(Ł) formulas

$$\Phi_1[P(\psi_1), \dots, P(\psi_k)] \text{ and } \Phi_2[P(\psi_1), \dots, P(\psi_k)]$$

find a **probabilistic substitution** τ such that

$$\tau(\Phi_1[P(\psi_1), \dots, P(\psi_k)]) = \tau(\Phi_2[P(\psi_1), \dots, P(\psi_k)])$$

Theorem (F., Ugolini - APAL'24)

he probabilistic unification problem for FP(Ł) is of **nullary** type.

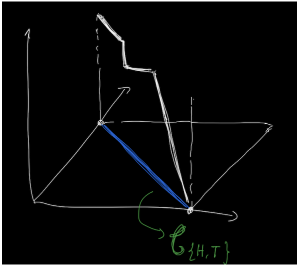
Funct. Representation

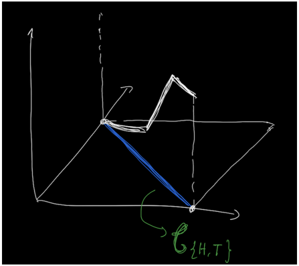
Theorem (F., Preto, Ugolini - KR'23)

An FP-formula $\Phi[P(\psi_1), \ldots, P(\psi_k)]$ is representable as the **restriction** of the McNaughton function

$$f_\Phi : [0,1]^k \to [0,1]$$

to the coherent set $\mathscr{C}_{\{\psi_1,\ldots,\psi_k\}}$.





JEP AND AMALGAMATION

Coherent MV-algebras satisfy the **joint embedding property**: given two coherent MV-algebras $\mathbf{A}_1$ and $\mathbf{A}_2$, there exists another coherent MV-algebra $\mathbf{B}$ such that both $\mathbf{A}_1$ and $\mathbf{A}_2$ embeds into $\mathbf{B}$.

What about:

- **Amalgamation**?
- **Strong** amalgamation?

Relevant for probability logic: to what property of coherence does "amalgamation" correspond?

- Any kind of **interpolation** property?

- If so, is it related to what de Finetti called the **Fundamental Theorem of Probability**?

Robinson consistency

Given two first order consistent theories T_1 and T_2 such that $T_1 \cap T_2$ is complete (in the common language of T_1 and T_2), then $T_1 \cup T_2$ is consistent.

It provides us with a notion of "compatibility" among first order theories:

Probabilistic version of Robinson consistency lemma

If $\beta_1 : \mathcal{E}_1 \to [0,1]$ and $\beta_2 : \mathcal{E}_2 \to [0,1]$ are coherent books on not necessarily equal sets of events $\mathcal{E}_1$ and $\mathcal{E}_2$, under which circumstances $\beta_1 \cup \beta_2$ is coherent?

Conclusion

Conclusion

- We have analyzed several semantics for the probability logic FP(Ł) by Hájek, Godo and Esteva
- We explored the geometric models for it
- We have introduced a class of algebraic models with respect to which FP(Ł) is *locally* sound and complete

- No AAL methods (bridge theorems) can be directly applied, however new techniques can be developed
- Unification; functional representation, JEP (known);
- Amalgamation, Robinson consistency ... (**future work**).

Conclusion

The logic FP(Ł) can be extended/expanded/modified so as to deal with:

- **Probability** of Łukasiewicz events: FP(Ł, Ł) and SFP(Ł, Ł) (with Godo and Montagna; Marra, Kroupa, Spada, Lapenta, Napolitano)
- **Belief functions** on classical events as in FP(S5,Ł) (Godo, Hájek and Esteva, 2001) or fuzzy events as in FP(KŁ$_n$,Ł) (with Godo and Marchioni; Dubois, Godo, Prade)
- **Probability** and **Belief functions** on **Belnap-Dunn** non-classical logic (Bilkova, Frittella, Kozhemiachenko, Majer; in a series of papers)
- **Possibilty** and **necessity** measures (several logic for this, still in collaboration with Godo, Marchioni et al.)
- **Ranking functions** (with Godo and Rosella).

Future work: what happens in the above cases?

Bibliography

1. T. Flaminio, S. Preto, S. Ugolini. **Reasoning about probability via continuous functions**. *Proceedings of KR'23*, 2023.
2. T. Flaminio, S. Ugolini. **Encoding de Finetti's coherence within Łukasiewicz logic and MV-algebras**. *Annals of Pure and Applied Logics*, 2024.
3. P. Hájek, L. Godo, F. Esteva. **Probability and Fuzzy Logic**. *Uncertainty in Artificial Intelligence UAI'95*, 1995.
4. V. Marra, L. Spada. **Duality, projectivity, and unification in Łukasiewicz logic and MV-algebras**. *Annals of Pure and Applied Logic*, 2013.
5. L. Zadeh. **Discussion: Probability Theory and Fuzzy Logic Are Complementary Rather Than Competitive**. *Technometrics* 37(3): 271–276, 1995.

thank you.

*

Fundamental Theorem of Probability

A similar question we can ask is what de Finetti called the "Fundamental Theorem of Probability" that firstly appeared in Boole's book and that reads as follows:

> *Given the probabilities of any events, of whatever kind, to find the probability of some other event connected with them.*

In a more formal setting (as stated by Hosni and Landes)

Key Problem [Hosni, Landes - '23]

Given a set of events Γ, find a quantification of the agent uncertainty on $\Delta \supseteq \Gamma$ subject to the following constraints:

- All available information is represented **faithfully**
- All the remaining uncertainty is quantified **coherently**

Here the question is how to **describe** such "minimal extension" algebraically by means of **coherent MV-algebras**.