Meaningful formalism and infinitary objects

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Constructive objects

Martin-Löf, *Notes on Constructive Mathematics* (1970):

> *All objects which we shall consider are to be constructive objects by which we mean finite configurations of signs. The signs, which can immediately be recognized as being equal or different, are treated as atoms that cannot be further decomposed.*

The term “constructive object” and its characterization were taken from Markov, “On constructive mathematics” (1962).

This is essentially Hilbert’s notion of *sign* from the “Neubegründung der Mathematik” (1922).

A privileged mode of givenness

Hilbert:

> *Als Vorbedingung für die Anwendung logischer Schlüsse und die Betätigung logischer Operationen muß vielmehr schon etwas in der Vorstellung gegeben sein: gewisse außerlogische diskrete Objekte, die anschaulich als unmittelbares Erlebnis vor allem Denken da sind.*

Idealization

Bernays (1930) was perhaps the first to note that a certain idealization is involved here.

There are limits to how large numbers

- can be physically realized as signs;
- can be recognized by us in a representation.

Bernays's solution was to postulate *formal abstraction* as the act of the mind through which constructive objects are given to us.

Formal objects

The notion of formal object reaches clarity in Kleene (1952) and Curry (1963).

- Inductively defined domain equipped with syntactic identity.

The inductive definition must be ordinary, not generalized.

Owing to the syntactic identity criterion, a formal object may be regarded as a *figure*.

For example, $2 + 2$ and 2×2 are different formal objects.

Examples of formal objects

Numbers as strings of ones:

$$\diamond : \mathbb{N} \qquad \frac{n : \mathbb{N}}{n1 : \mathbb{N}}$$

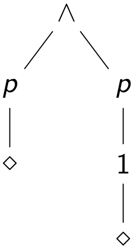
Formulas in a language for classical propositional logic:

$$\frac{n : \mathbb{N}}{p_n : \mathsf{F}} \qquad \frac{\varphi : \mathsf{F}}{\neg\varphi : \mathsf{F}} \qquad \frac{\varphi : \mathsf{F} \qquad \psi : \mathsf{F}}{\varphi \wedge \psi : \mathsf{F}}$$

Finitary nature of formal objects

A formal object is completely determined by its construction history, which may be displayed as a finite tree.

For example, the formula $p_\diamond \wedge p_{\diamond 1}$ may be associated with the following tree:



Infinitary objects

The constructive second number class:

$$\diamond : \mathcal{O} \qquad \frac{n : \mathcal{O}}{n1 : \mathcal{O}} \qquad \frac{f : \mathbb{N} \to \mathcal{O}}{\sup f : \mathcal{O}}$$

Let $i : \mathbb{N} \to \mathcal{O}$ be the embedding $i(n) = n$ and define

$$\omega := \sup i$$

The object ω has a finite construction history,

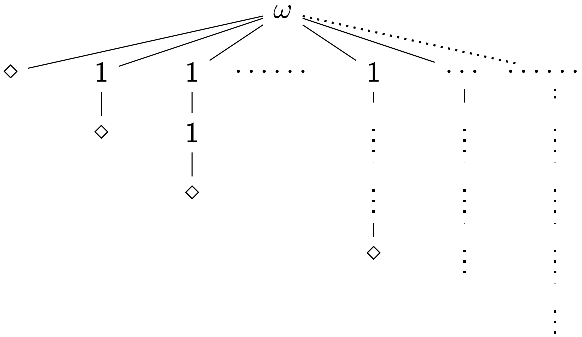
sup

i

which does not, however, capture the essence of ω.

The infinity of ω

An infinite tree would be needed to show that ω "depends on" $\diamond, \diamond 1, \diamond 11, \ldots$



Form, content, and infinity

A mathematical object is not a mere form – it also has content.

Every object has finite form.

An infinitary object has infinite content.

The content of an object a can be displayed in a tree, $\mathcal{T}(a)$, that is different from the construction history of a.

That a has infinite content just means that $\mathcal{T}(a)$ is infinite.

Meaningful formalism

Meaningful formalism is the view that a mathematical object just is a meaningful expression.

Martin-Löf, "The Hilbert–Brouwer controversy resolved?" (2008):

> *the mathematical objects are not just purely formal sign configurations: they are meaningful sign configurations, and that is what gives them properties which do not come from their combinatorial nature, but come from the meaning with which they are endowed, as beautifully stated by Gödel in the beginning of his Dialectica paper.*

Merits of meaningful formalism

Mathematical objects are present to us as signs, but they also have a conceptual side.

- Sensibility and understanding both contribute to our knowledge of mathematical objects.

No need to postulate formal abstraction or rational intuition.

- We grasp mathematical objects by means of the same capacity through which we understand language.

Axioms and rules are seen to be evident on the basis of our understanding of mathematical language.

Constructive type theory

Type structure:

- Ground types
    1. Inductively defined types
    2. The type set of inductively defined types
- Function types, $\alpha \to \beta$

Modes of object formation:

- Formation rules
- Introduction rules (= clauses in a fundamental inductive definition)
- Explicit definition
- Definition of a function by structural induction, possibly involving recursion (= elimination rules)
- Functional abstraction

Examples

Formation rule

$$\mathbb{N} : \mathsf{set}$$

Introduction rules

$$0 : \mathbb{N} \qquad \frac{n : \mathbb{N}}{\mathsf{s}(n) : \mathbb{N}}$$

Explicit definition

$$1 := \mathsf{s}(0)$$

Definition by structural induction, involving recursion

$$\begin{aligned} m + 0 &:= m \\ m + \mathsf{s}(n) &:= \mathsf{s}(m + n) \end{aligned}$$

Evaluation

Evaluation, or computation, plays the role of reference (also in Dummett, Moschovakis, and Tichý).

Evaluation transforms one meaningful expression (qua expression) into another.

Reference in this sense therefore does not take us outside of the mathematical language.

An object/expression that may serve as reference is said to be in canonical form.

$$\left.\begin{array}{rl} 2+2 & \Longrightarrow \\ 2\times 2 & \Longrightarrow \end{array}\right\}\mathsf{s}(\mathsf{s}(\mathsf{s}(\mathsf{s}(0))))$$

$$\text{object} \quad \Longrightarrow \quad \text{object in canonical form}$$

Evaluation in the presence of infinitary objects

A higher-order object such as $i : \mathbb{N} \to \mathcal{O}$ is not itself computed/evaluated.

Since $\sup i$ is constructed by the application of sup to i, we cannot require that evaluation follows the applicative order, where all arguments, $a_1, \ldots, a_n$, are evaluated before $f(a_1, \ldots, a_n)$ is evaluated.

Instead, evaluation is *lazy*: whether an object is canonical depends only on its outermost form.

- Canonical element of $\mathcal{O}$: $\sup f$ for any $f : \mathbb{N} \to \mathcal{O}$.
- Canonical element of $\mathbb{N}$: 0 or $\mathsf{s}(n)$ for any $n : \mathbb{N}$.

Content analysis

We can spell out the content of an object through analysis.

Three procedures:

- Ground type
  - Evaluation to canonical form
  - Extraction of arguments from canonical forms
- Function type
  - Function application

The tree $\mathcal{T}(a)$

The tree $\mathcal{T}(a)$ exhibits the result of the analysis of a.

It is defined by the following two construction principles:

- a is of ground type.
  - Evaluate a to canonical form $c(b_1, \ldots, b_n)$.
  - Label the root of $\mathcal{T}(a)$ by c.
  - Let the immediate subtrees of $\mathcal{T}(a)$ be $\mathcal{T}(b_1), \ldots, \mathcal{T}(b_n)$.
- a is of function type, $B \to C$.
  - Label the root of $\mathcal{T}(a)$ by a.
  - Let the immediate subtrees of $\mathcal{T}(a)$ be every $\mathcal{T}(a(b))$, as b ranges over B.

First example. The tree $\mathcal{T}(2+2)$

$$2+2 \Longrightarrow \mathsf{s}(2+1)$$

$$2+1 \Longrightarrow \mathsf{s}(2+0)$$

$$2+0 \Longrightarrow \mathsf{s}(1)$$

$$1 \Longrightarrow \mathsf{s}(0)$$

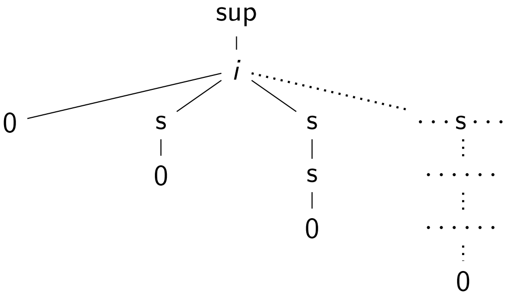
$$0 \Longrightarrow 0$$

Second example. The tree $\mathcal{T}(2 \times 2)$

$$\begin{aligned} 2 \times 2 &\Longrightarrow \mathsf{s}(2 \times 1 + 1) \\ 2 \times 1 + 1 &\Longrightarrow \mathsf{s}(2 \times 1) \\ 2 \times 1 &\Longrightarrow \mathsf{s}(2 \times 0 + 1) \\ 2 \times 0 + 1 &\Longrightarrow \mathsf{s}(2 \times 0 + 0) \\ 2 \times 0 &\Longrightarrow 0 \end{aligned}$$

Third example. The tree $\mathcal{T}(\omega)$

- Since $\omega \Rightarrow \sup i$, the root of the tree $\mathcal{T}(\omega)$ is labelled sup.
- The immediate subtree of $\mathcal{T}(\omega)$ is $\mathcal{T}(i)$.
- The immediate subtrees of $\mathcal{T}(i)$ are all the trees $\mathcal{T}(i(n))$ as n ranges over $\mathbb{N}$.



Infinity of $\mathcal{T}(\omega)$

The infinity of $\mathcal{T}(\omega)$ is potential.

It is constructed branch by branch, not rank by rank.

Once an argument $n : \mathbb{N}$ is provided, the subtree $\mathcal{T}(i(n))$ is constructed.

We may think of $\mathcal{T}(a)$ as a picture of the process of analyzing a.

That a is an infinitary object just means that its analysis is an infinite, and therefore incompletable, process.