Deciding equations in conuclear residuated lattices

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Just like residuated lattices are substructural generalizations of Boolean algebras, a **conucleus** is a substructural version of an S4 modal operator.

(We consider **only** the box operator and **only** algebraic semantics here.)

MAIN RESULT

Some S4-like substructural modal logics, in particular some versions of S4 Łukasiewicz and Abelian logic, have a decidable deducibility problem.

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The variety of integral commutative **cancellative** residuated lattices has a decidable universal theory, as does the subvariety of fully distributive ones.

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THEOREM (Horčík 2006)

The subvariety of **semilinear** CanICRLs has a decidable universal theory.

What are conuclei?

A **(bounded) residuated lattice** is an algebra with:

- lattice operations $x \wedge y$ and $x \vee y$
- (lattice top and bottom $\top$ and $\bot$)
- monoidal multiplication $x \cdot y$ with unit 1
- two division operations $x \backslash y$ and x / y

such that the **residuation law** holds:

$$y \leq x \backslash z \iff x \cdot y \leq z \iff x \leq z / y.$$

Integral: $1 = \top$ **Commutative**: $x \cdot y = y \cdot x$

IRL = integral RL **ICRL** = integral commutative RL

EXAMPLE. Lattice-ordered groups (ℓ-groups):

$$y \leq x^{-1} z \iff x \cdot y \leq z \iff x \leq z y^{-1}.$$

EXAMPLE. Boolean and Heyting algebras:

$$y \leq x \to z \iff x \wedge y \leq z \iff x \leq z \leftarrow y.$$

A **conucleus** on a RL $\mathbf{A}$ is an interior operator $\Box$ on $\mathbf{A}$ whose image

$$\mathbf{A}_{\Box} := \{a \in \mathbf{A} \mid \Box a = a\} = \Box[\mathbf{A}]$$

is a submonoid of $\mathbf{A}$. Equivalently, $\Box$ is an order-preserving operation with

$$\Box\Box x = \Box x \leq x, \qquad \Box x \cdot \Box y \leq \Box(x \cdot y), \qquad 1 \leq \Box 1.$$

Cf. the literature on Hájek's "very true" operator (2001).

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An algebra of the form $\langle \mathbf{A}, \Box \rangle$ will be called **conuclear residuated lattice**.

$$\mathbb{C}\mathrm{x}(\mathsf{K}) := \{\langle \mathbf{A}, \Box \rangle \mid \mathbf{A} \in \mathsf{K} \text{ and } \Box \text{ is a conucleus on } \mathbf{A}\}.$$

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Example. $\mathbb{C}\mathrm{x}$(Boolean algebras) = S4 modal algebras.

Example. The map $\Box\colon x \mapsto 1 \wedge x$ is always a conucleus.

$\mathbf{A}_\Box$ is a residuated lattice in its own right, but it is not a subalgebra of $\mathbf{A}$.

The operations $\vee$, $\cdot$, 1 of $\mathbf{A}_\Box$ agree with $\mathbf{A}$. Others operations need not:

$$x \backslash^{\mathbf{A}_\Box} y := \Box^{\mathbf{A}}(x \backslash^{\mathbf{A}} y), \quad x /^{\mathbf{A}_\Box} y := \Box^{\mathbf{A}}(x /^{\mathbf{A}} y), \quad x \wedge^{\mathbf{A}_\Box} y := \Box^{\mathbf{A}}(x \wedge^{\mathbf{A}} y).$$

The residuated lattice $\mathbf{A}_\Box$ will be called a **conuclear image** of $\mathbf{A}$ (w.r.t. $\Box$).

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Example. $\mathbb{C}$(Boolean algebras) = Heyting algebras.

(This is the Gödel translation between S4 and intuitionistic logic.)

Example. The **negative cone** of $\mathbf{A}$ is the IRL $\mathbf{A}^- := \mathbf{A}_\Box$ for $\Box\colon x \mapsto 1 \wedge x$.

$\wedge$**-conuclei**, $\vee$**-conuclei** and $\odot$**-conuclei** satisfy, respectively:

$$\Box(x \wedge y) = \Box x \wedge \Box y \qquad \Box(x \vee y) = \Box x \vee \Box y \qquad \Box(x \cdot y) = \Box x \cdot \Box y$$

This gives us finer-grained control over which equations are preserved.

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All **conuclei:** preserve equations and quasi-equations in $\{\vee, \cdot, 1\}$:

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$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

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$\vee$**-conuclei:** preserve **semilinearity** (in ICRLs):

$$(x \backslash y) \vee (y \backslash x) = 1$$

$\odot$**-conuclei:** preserve **divisibility** (in IRLs):

$$x \cdot (x \backslash y) = x \wedge y = (y / x) \cdot x$$

Montagna and Tsinakis (2010)

- $\mathbb{C}$(Abelian ℓ-groups) $=$ cancellative CRLs.
- $\mathbb{C}_{\wedge}$(Abelian ℓ-groups) $=$ fully distributive cancellative CRLs.
- $\mathbb{C}_{\vee}$(Abelian ℓ-groups) $=$ semilinear cancellative CRLs.

A residuated lattice is **cancellative** if it satisfies the quasi-equations

$$z \cdot x = z \cdot y \implies x = y, \qquad x \cdot z = y \cdot z \implies x = y,$$

or equivalently the equations $zx\backslash zy = x\backslash y$, $\quad xz/yz = x/y$.

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- $\mathbb{C}$(Abelian ℓ-group cones) $=$ cancellative ICRLs.
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An **Abelian ℓ-group cone** is the negative cone of an Abelian ℓ-group.

Theorem (Bahls, Cole, Galatos, Jipsen, Tsinakis 2003)

Abelian ℓ-group cones are precisely the cancellative divisible ICRLs.

Decidability results

PROBLEM. Given a variety V of integral residuated lattices:

Does $\mathbb{C}\mathrm{x}(\mathsf{V})$ have a decidable quasi-equational theory?

Equivalently, does $\mathbb{C}\mathrm{x}(\mathsf{V})$ have a decidable universal theory?

STRATEGY.

Find a manageable generating class for conuclear V-algebras.

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For example, $\mathbb{C}(\mathsf{V}) \models (x\backslash y) \vee (y\backslash x) = 1 \iff \mathbb{C}\mathrm{x}(\mathsf{V}) \models \Box(x\backslash y) \vee \Box(y\backslash x) = 1.$

Decidability theorem

The following varieties have a decidable universal theory:

- MV-algebras with a conucleus ($\wedge$-conucleus, $\vee$-conucleus).
- Abelian ℓ-group cones with a conucleus ($\wedge$-conucleus, $\vee$-conucleus).
- Abelian ℓ-groups with a negative conucleus ($\wedge$-conucleus, $\vee$-conucleus).

So do the universal classes of totally ordered algebras in these varieties.

A conucleus $\Box$ on $\mathbf{A}$ is **negative** if $\Box x \leq 1$, i.e. if $\mathbf{A}_{\Box} \subseteq \mathbf{A}^{-}$.

The results for Abelian ℓ-groups and MV-algebras follow from the result for Abelian ℓ-group cones. We will therefore focus on the cone case only.

The Montagna–Tsinakis representation immediately yields the following.

COROLLARY

The following varieties have a decidable universal theory:

- Cancellative ICRLs.
- Fully distributive cancellative ICRLs.
- Semilinear cancellative ICRLs.

So do totally ordered cancellative ICRLs.

The first case was a long-standing open problem, see Bahls et al. (2003).

The totally ordered and semilinear case were obtained by Horčík (2006).

Conuclei and partial embeddings

A **universal class** of algebras is axiomatized by universal sentences, e.g.:

$$\forall xy\,(x \leq y \text{ or } y \leq x).$$

Theorem. The universal class **generated** by K is $\mathbb{U}(\mathsf{K}) := \mathbb{I}\mathbb{S}\mathbb{P}_{\mathrm{U}}(\mathsf{K})$.

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Theorem. The universal class **generated** by K is $\mathbb{U}(\mathsf{K}) := \mathbb{ISP}_{\mathrm{U}}(\mathsf{K})$.

Universal classes and partial embeddings (Maltsev 1973)

Let $\mathbf{A}$ be an algebra and K a class of algebras. Then $\mathbf{A} \in \mathbb{U}(\mathsf{K}) := \mathbb{ISP}_{\mathrm{U}}(\mathsf{K})$ if and only if for each finite set $X \subseteq \mathbf{A}$ the finite partial algebra $\mathbf{A}|_X$ has an embedding ι into some $\mathbf{B} \in \mathsf{K}$. (We say that **A locally embeds** into K.)

More explicitly, for each finite $X \subseteq \mathbf{A}$ there is an injective map $\iota : X \hookrightarrow \mathbf{B}$ such that for each n-ary primitive operation $\circ$ and all $a_1, \ldots, a_n, b \in X$

$$\circ^{\mathbf{A}}(a_1, \ldots, a_n) = b \implies \circ^{\mathbf{B}}(\iota(a_1), \ldots, \iota(a_n)) = \iota(b).$$

A class K has the **Finite Embeddability Property** (**FEP**) if each K-algebra locally embeds into a finite K-algebra, or equivalently if $\mathsf{K} \subseteq \mathbb{U}(\mathsf{K}_{\text{fin}})$.

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FEP and decidability (Evans 1969)

If a universal class is finitely axiomatizable and enjoys the FEP (i.e. is generated by finite algebras), then it has a decidable universal theory.

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FEP for IRLs (Blok and van Alten 2002 and 2005)

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The varieties of integral RLs and integral commutative RLs have the FEP.

FEP for subvarieties of IRLs (Galatos and Jipsen 2013)

Each subvariety of integral residuated lattices axiomatized by some set of equations in the signature $\{\vee, \cdot, 1\}$ has the FEP.

Proof uses residuated frames.

Universal classes

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Universal classes of conuclear IRLs

$\mathbf{A} \in \mathsf{IRL}$ locally π-embeds into $\mathsf{K} = \mathbb{S}(\mathsf{K}) \subseteq \mathsf{IRL} \implies \mathbb{C}\mathrm{x}(\mathbf{A}) \in \mathbb{U}\mathbb{C}\mathrm{x}(\mathsf{K})$

Varieties of algebras

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Varieties of conuclear IRL**s**

$\mathsf{V} \subseteq \mathsf{IRL}$ locally π-embeds into $\mathsf{K} = \mathbb{S}(\mathsf{K}) \subseteq \mathsf{V} \implies \mathbb{C}\mathrm{x}(\mathsf{V}) = \mathbb{U}\mathbb{C}\mathrm{x}(\mathsf{K})$

Given an integral RL $\mathbf{A}$ and a **finite** set $S \subseteq \mathbf{A}$, let

$$\begin{aligned}\langle S\rangle^{\mathbf{A}} &:= \text{the } \{\vee, \cdot, 1\}\text{-subalgebra of } \mathbf{A} \text{ generated by } S. \\ &= \{p_1 \vee \cdots \vee p_n \mid p_1, \ldots, p_n \text{ are products of elements from } S.\}\end{aligned}$$

$\langle S\rangle^{\mathbf{A}}$ is the image of a conucleus $\Box_S$ on $\mathbf{A}$ (by Higman's Lemma).

(This is a small lie: $\Box_S a$ is undefined if there is no $x \in \langle S\rangle^{\mathbf{A}}$ below a.)

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Crucially, each conucleus **locally** looks like some $\Box_S$.

FACT. Given a conucleus $\Box$ on an integral RL $\mathbf{A}$ and finite $X \subseteq \mathbf{A}$,

$$\langle \mathbf{A}, \Box \rangle|_X = \langle \mathbf{A}, \Box_S \rangle|_X \qquad \text{for } S := X \cap \mathbf{A}_\Box.$$

A poset is said to be **dually well partially ordered** if one of the following equivalent conditions holds:

- each downset is finitely generated,
- there are no infinite antichains and no infinite ascending chains.
- for each sequence $(a_n)_{n\in\omega}$ there are $i \leq j$ with $a_i \geq a_j$.

HIGMAN'S LEMMA (1952)

Let **A** be an algebra in a finite signature and let $\leq$ be a partial order on **A** such that for each n-ary primitivate operation $\circ$

- $a_1 \leq b_1, \ldots, a_n \leq b_n \implies \circ^{\mathbf{A}}(a_1, \ldots, a_n) \leq \circ^{\mathbf{A}}(b_1, \ldots, b_n)$,
- $\circ(a_1, \ldots, a_n) \leq a_i$ for each $i \in \{1, \ldots, n\}$.

If **A** is generated by a dually well partially ordered subset (for example, a finite subset), then **A** itself is dually well partially ordered.

(This is the special case of Higman's Lemma for finite signatures.)

Consider $\mathbf{A}, \mathbf{B} \in \mathsf{IRL}$, a finite set $X \subseteq \mathbf{A}$, and an embedding $\iota \colon \mathbf{A}|_X \hookrightarrow \mathbf{B}$. This embedding is called a **π-embedding** if for all $x_1, \ldots, x_n, y \in X$

$$x_1 \cdot \ldots \cdot x_n \leq^{\mathbf{A}} y \iff \iota(x_1) \cdot \ldots \cdot \iota(x_n) \leq^{\mathbf{B}} \iota(y)$$

Theorem. If $\mathbf{A} \in \mathsf{IRL}$ locally π-embeds into a class $\mathsf{K} = \mathbb{S}(\mathsf{K}) \subseteq \mathsf{IRL}$, then

$$\mathbb{C}\mathrm{x}(\mathbf{A}) \subseteq \mathbb{U}\mathbb{C}\mathrm{x}(\mathsf{K}).$$

Proof. We may restrict to conuclei of form $\Box_S$ with $S \subseteq X$. Then

$$\pi\text{-embedding } \iota \colon \mathbf{A}|_X \hookrightarrow \mathbf{B} \implies \text{embedding } \iota \colon \langle \mathbf{A}, \Box_S \rangle|_X \hookrightarrow \langle \mathbf{B}, \Box_{\iota[S]} \rangle.$$

Corollary. If each finite partial subalgebra of each algebra in a universal class $\mathsf{K} \subseteq \mathsf{IRL}$ π-embeds into a finite K-algebra, then $\mathbb{C}\mathrm{x}(\mathsf{K})$ has the FEP.

CONUCLEAR K-ALGEBRAS INHERIT THE FEP

Let $\mathsf{K} \subseteq \mathsf{IRL}$ be a universal class with locally finite semigroup reducts. (For example, commutative and $x^n = x^{n+1}$.) Then:

$$\mathsf{K} \text{ has the FEP} \implies \mathbb{C}\mathrm{x}(\mathsf{K}) \text{ has the FEP.}$$

The above is an easy but useful consequence of the π-embedding theorem.

TRIVIAL. If $\mathbb{C}\mathrm{x}(\mathsf{K})$ has the FEP, then $\mathbb{C}(\mathsf{K})$ has the FEP.

Conuclear K-algebras inherit the FEP

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- Boolean algebras have the FEP because they are locally finite.

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- So S4 modal algebras have the FEP (McKinsey 1941).

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- So Heyting algebras have the FEP (McKinsey & Tarski 1946).

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- So S4 modal algebras have the FEP (McKinsey 1941).
- So Heyting algebras have the FEP (McKinsey & Tarski 1946).
- So conuclear Heyting algebras have the FEP (de Groot & Shillito ’25).

The methodology of Blok and van Alten applies to conuclear IRLs.

THEOREM (Amano 2006)
Conuclear $\mathbf{FL_{ew}}$-algebras (conuclear bounded ICRLs) have the FEP.

OPEN PROBLEM.
Does each variety of conuclear integral RLs axiomatized by some set of equations in $\{\vee, \cdot, 1\}$ has the FEP (cf. the result of Galatos and Jipsen)?

The case of Abelian ℓ-group cones

Varieties of conuclear IRLs

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Application to Abelian ℓ-groups

Totally ordered Abelian groups locally ω-embed into finite lex-powers of $\mathbb{Z}$.

Consider $\mathbf{A}, \mathbf{B} \in \mathsf{ICRL}$, a finite set $X \subseteq \mathbf{A}$, and an embedding $\iota : \mathbf{A}|_X \hookrightarrow \mathbf{B}$. This embedding is called a ω-**embedding** if for all $x, y \in X$

$$x^n \nleq^{\mathbf{A}} y \text{ for all } n \in \mathbb{N} \implies \iota(x)^n \nleq^{\mathbf{B}} \iota(y) \text{ for all } n \in \mathbb{N}.$$

One can think of this as preserving the relation of being infinitely below:

$$y \ll^{\mathbf{A}} x \implies \iota(y) \ll^{\mathbf{B}} \iota(x).$$

(Consider a negative infinitesimal $-\varepsilon$. Then $-1 \ll -\varepsilon$.)

Lemma (ω-embeddings can replace π-embeddings).

The following are equivalent for each $\mathbf{A} \in \mathsf{ICRL}$ and $\mathsf{K} \subseteq \mathsf{ICRL}$:

- Each finite partial subalgebra $\mathbf{A}|_X$ has a π-embedding into K.
- Each finite partial subalgebra $\mathbf{A}|_X$ has an ω-embedding into K.

Proof uses Higman's Lemma (in a more complicated way).

Let V_{fsi} denote the class of finitely subdirectly irreducible V-algebras.

Example. If V = Abelian ℓ-groups, V_{fsi} = totally ordered Abelian groups.

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Theorem (reduction to f.s.i. algebras).

Let $\mathsf{V} \subseteq \mathsf{ICRL}$ be a variety. If $\mathsf{V}_{\mathrm{fsi}}$ locally ω-embeds into $\mathsf{K} \subseteq \mathsf{V}$, then

$$\mathbb{C}\mathrm{x}(\mathsf{V}) = \mathbb{U}(\mathbb{C}\mathrm{x}\mathbb{S}\mathbb{P}_{\mathrm{fin}}(\mathsf{K})).$$

Corollary. In particular, for each variety $\mathsf{V} \subseteq \mathsf{ICRL}$

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Example. If $\mathsf{V}_{\mathrm{fsi}}$-algebras are totally ordered, then conuclear V-algebras are generated as a universal class by conuclear V-algebras of finite width.

HAHN'S REPRESENTATION (1907)

Each totally ordered Abelian group embeds into a lexicographic power of $\mathbb{R}$.

For example, $\mathbb{R}^2_{\mathrm{lex}} := \mathbb{R} \times_{\mathrm{lex}} \mathbb{R}$ is the Abelian group $\mathbb{R} \times \mathbb{R}$ ordered as:

$$\langle a, b \rangle \leq \langle c, d \rangle \iff \text{either } a < c \text{ or } (a = c \text{ and } b \leq d).$$

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In each totally ordered Abelian group $\mathbf{G}$, for $a, b \in \mathbf{G}^-$ we have

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Lemma. Each totally ordered Abelian group cone locally ω-embeds into $(\mathbb{Z}^n_{\text{lex}})^-$ for some n, which in turn locally ω-embeds into $(\mathbb{Z}^n)^-$.

Here $\mathbb{L}\mathrm{ex}_{\mathrm{fin}}(\mathbb{Z})$ denotes the class of finite lexicographic powers of $\mathbb{Z}$.

Generation results for conuclear algebras

The **universal class** of

- conuclear Abelian ℓ-group cones is generated by $\mathbb{C}\mathrm{x}(\mathbb{P}_{\mathrm{fin}}(\mathbb{Z}^-))$.
- $\wedge$-conuclear Abelian ℓ-group cones is generated by $\mathbb{C}\mathrm{x}_\wedge(\mathbb{P}_{\mathrm{fin}}(\mathbb{Z}^-))$.
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Horčík (2006) does not **explicitly** discuss conuclei, but his methods would probably have sufficed to handle the totally ordered and $\vee$-conuclear case.

FEP for conuclear MV-algebras

- The variety of MV-algebras with a conucleus has the FEP.
- The variety of MV-algebras with a $\wedge$-conucleus has the FEP.
- The variety of MV-algebras with a $\vee$-conucleus **does not** have the FEP.
- Totally ordered conuclear MV-algebras **do not** have the FEP.

(In the totally ordered case, lexicographic products are necessary. Outside of the totally ordered case, these can be simulated by ordinary products.)

Open problems

Arguably $\odot$-conuclei are more interesting as S4 modalities on MV-algebras.

$\odot$-conuclei satisfy $\Box(x \cdot y) = \Box x \cdot \Box y$.

Follows from Jipsen and Montagna (2010)

- $\mathbb{SC}_{\odot}$(MV-algebras) = divisible bounded ICRLs.

These conuclei come from the representation of divisible bounded ICRLs as subalgebras of **poset products** of MV-chains. Compare: each Heyting algebra embeds into a poset product of (two-element) Boolean chains.

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Open problem.
Decidability for $\odot$-conuclear MV-algebras or $\odot\wedge$-conuclear MV-algebras. (Is this the equational theory of poset product conuclei?)

OPEN PROBLEM.
Describe $\mathbb{C}(\mathsf{MV})$ or $\mathbb{C}_\wedge(\mathsf{MV})$ or $\mathbb{C}_\vee(\mathsf{MV})$ or $\mathbb{C}_\odot(\mathsf{MV})$.

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Looking at the lattice of varieties of RLs closed under $\mathbb{C}$ ($\mathbb{C}_\wedge, \dots$) will tell us more about the **large-scale structure** of the lattice of varieties of RLs.

The large-scale structure may be more tractable than the fine structure: e.g. the lattice of varieties of Heyting algebras collapses to a single point.

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OPEN PROBLEM.
Describe the lattice of varieties of divisible ICRLs closed under $\odot$-conuclei.

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Determine the computational complexity of the (quasi)equational theories of conuclear MV-algebras, conuclear Abelian ℓ-group cones, etc.

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Thank you for your attention!