Intermediate quantifiers and their syllogisms in fuzzy natural logic

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Natural logic

George Lakoff: *Linguistic and Natural Logic, Synthese 22, 1970*

Natural logic is a collection of terms and rules that come with natural language that allows us to reason and argue in it

Hypothesis (Lakoff)

Natural language employs a relatively small finite number of atomic predicates that take sentential complements (sentential operators) and are related to each other by meaning-postulates that do not vary from language to language.

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Fuzzy Natural Logic (FNL)

FNL is:

- A system of formal theories of mathematical fuzzy logic.
  So far:
  - Theory of evaluative linguistic expressions
    (*small, very small, medium, large, etc.*)
  - Theory of fuzzy and intermediate quantifiers
    (*most, a lot of, few, many, etc.*)

    and generalized Aristotle's syllogisms
  - Theory of fuzzy/linguistic IF-THEN rules and logical inference
    (*Perception-based Logical Deduction*)
- It requires a mathematical model of linguistic semantics
  Fuzzy Type Theory (Higher-order Fuzzy Logic)

Ideas of P. Materna and M. Duži are incorporated

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Fuzzy type theory — higher-order fuzzy logic

Higher-order logic (type theory) is a necessary tool in formal models of linguistic semantics

B. Russel, A. Church, L. Henkin, P. Andrews

Fuzzy type theory is an extended lambda calculus:

- more logical axioms
- many-valued semantics

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Types

Elementary types: o (truth values), ϵ (objects)

Composed types: $\beta\alpha$

Formulas have types: $A_\alpha \in Form_\alpha$, $\lambda x_\alpha\, C_\beta$

Formulas of type o are propositions

Truth values form an MV_Δ-algebra

Standard Łukasiewicz MV_Δ-algebra

$$\mathscr{E} = \langle [0,1], \vee, \wedge, \otimes, \Delta, \rightarrow, 0, 1\rangle$$

$\vee, \wedge =$ minimum, maximum

$a \otimes b = 0 \vee (a + b - 1)$ (Łukasiewicz conjunction)

$a \rightarrow b = 1 \wedge (1 - a + b)$ (Łukasiewicz implication)

$\neg a = a \rightarrow 0 = 1 - a$

$$\Delta(a) = \begin{cases} 1 & \text{if } a = 1, \\ 0 & \text{otherwise.} \end{cases}$$

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Fuzzy type theory — higher-order fuzzy logic

Frame: a system of sets with fuzzy equality

$$\mathscr{M} = \langle (M_\alpha, \mathring{=}_\alpha)_{\alpha \in Types}, \mathscr{E} \rangle$$

Interpretation of formulas

$\mathscr{M}(A_o) \in M_o = [0, 1]$ $\quad$ $\mathscr{M}(A_\epsilon) \in M_\epsilon$

$\mathscr{M}(A_{\beta\alpha}) : M_\alpha \longrightarrow M_\beta;$ $\quad$ fuzzy set $\mathscr{M}(A_{o\alpha}) \underset{\sim}{\subset} M_\alpha$

$\mathscr{M}(\lambda x_\alpha\, C_\beta) : M_\alpha \longrightarrow M_\beta$

Standard model $M_{\beta\alpha} = M_\beta^{M_\alpha}$ $\quad$ General model $M_{\beta\alpha} \subseteq M_\beta^{M_\alpha}$

Theorem (Completeness)

(a) *A theory T of FTT is consistent iff it has a general model $\mathscr{M}$.*

(b) $T \vdash A_o$ *iff* $T \models A_o$
holds for every theory T and a formula A_o

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Generalized (fuzzy) quantifiers

A. Mostowski (1957), P. Lindström (1966), R. Montague, J. Van Benthem, D. Westerståhl, J. Barwise, L. A. Zadeh

Quantifiers in natural language

Words or expressions that precede and modify nouns. They specify quantity of elements in a domain of discourse bearing some property.

- *A few beautiful roses are standing on wooden table*
- *Many red flowers are damaged by strong wind*

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Generalized fuzzy quantifier of type $\langle k_1, \dots, k_n \rangle$

A function $\mathbf{Q}$ with the symbol Q which to each set M assigns a fuzzy relation

$$\mathbf{Q}_M \subsetapprox \mathscr{F}(M^{k_1}) \times \cdots \times \mathscr{F}(M^{k_n})$$

such that

$$\mathscr{M}((Q\mathbf{x}_1 \cdots \mathbf{x}_n)(A_1, \dots, A_n)) = a \quad \text{iff}$$
$$\mathbf{Q}_M(\langle \mathrm{Sat}_{\mathscr{M}}(A_1, \mathbf{x}_1), \dots, \mathrm{Sat}_{\mathscr{M}}(A_n, \mathbf{x}_n) \rangle) = a.$$

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Quantifiers in mathematical fuzzy logic

Example (Universal quantifier $\forall$)

Functional $\forall_M$ together with the symbol $\forall$ assigning to each set M a fuzzy set

$$\forall_M = \left\{ \bigwedge_{m\in M} H(m) \big/ H \;\middle|\; H \subsetsim M \right\}$$

so that

$$\mathscr{M}((\forall x)A) = a \quad \text{iff} \quad \forall_M(\mathrm{Sat}_{\mathscr{M}}(A, x)) = a.$$

This is equivalent with the interpretation of a universally quantified formula in mathematical fuzzy logic

$$\mathscr{M}((\forall x)A) = \bigwedge_{m\in M} \mathscr{M}(A_x[m]).$$

Intermediate quantifiers as special fuzzy quantifiers

Example (Intermediate quantifiers)

All, Most, Almost all, Many, A lot of, Some, A few, Several, A little

Most women in the party have long hair

Several dry trees are very high

Quantifiers of type $\langle 1, 1\rangle$

Q B are A

Many (Q)

thick books on the bookshelf (B)

are

very interesting (A)

P. L. Peterson, *Intermediate Quantifiers. Logic, linguistics, and Aristotelian semantics*, Ashgate, Aldershot 2000

Intermediate quantifiers as special fuzzy quantifiers

Size of (fuzzy) sets

Measure: a natural way how size of a (fuzzy) set can be characterized

Interpretation of intermediate quantifiers

Classical quantifiers $\forall$ and $\exists$ taken over a *smaller* class of elements:
Size of this class is:

- characterized by measure,
- evaluated by an appropriate evaluative linguistic expression.

Formal theory of intermediate quantifiers: a special theory in Łukasiewicz fuzzy type theory (ŁFTT)

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Measure of a fuzzy set

A function $\mu : \mathscr{F}(M) \times \mathscr{F}(M) \longrightarrow [0, 1]$
(determined axiomatically; 4 axioms)

$\mu(B, A)$: measure of $A \subsetneqq M$ w.r.t. $B \subsetneqq M$

Example (measure)

Let M be finite, $A, B \subsetneqq M$. Put

$$|A| = \sum_{m \in M} A(m).$$

Then

$$\mu(B, A) = \begin{cases} 1 & \text{if } B \subseteq A, \\ \min\left\{1, \frac{|A|}{|B|}\right\} & \text{if } A \subseteq B \\ 0 & \text{otherwise} \end{cases}$$

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Evaluation using natural language expressions

Evaluative linguistic expressions

Special expressions of natural language using which people evaluate encountered phenomena and processes

Example

Small, Medium, Big

very short, rather strong, more or less medium, roughly big, extremely high, very intelligent, significantly important, etc.

Syntactical structure of simple evaluative expressions

$\langle$Hedge$\rangle\langle$TE-adjective$\rangle$

very small

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Evaluative linguistic expression $\mathscr{A}$

Context $w = [v_L, v_S] \cup [v_S, v_R]$

Intension of $\mathscr{A}$: $W \longrightarrow \mathscr{F}(M)$

Extension of $\mathscr{A}$ in a context w

[width=110mm]/Users/Vilem/GoogleDrive/Clanky/Archiv/Obrazky//EvExpr
Crop.pdf

w= $[0, 4] \cup [4, 10]$

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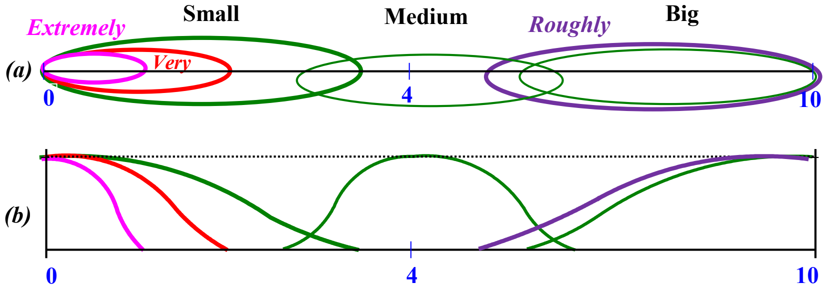

w= $[0, 4] \cup [4, 10]$

Cut of a fuzzy set

$$(B|Z)(m) = \begin{cases} B(m), & \text{if } B(m) = Z(m), \\ 0 & \text{otherwise.} \end{cases}$$

Example

$M = \{a, b, c, d, e, f, g\}$

$$B = \{^{0.1}/a, ^{1}/b, ^{0.7}/c, \ ^{0.4}/d, ^{0.2}/f, ^{0.9}/g\}$$
$$Z = \{^{0.1}/a, ^{1}/b, ^{0.3}/c, \ ^{0.4}/d\}$$

$$B|Z = \{^{0.1}/a, ^{1}/b, \qquad ^{0.4}/d\}$$

The original membership degrees in B are not modified in $B|Z$.

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Formal theory of intermediate quantifiers

- Intermediate quantifiers are specific formulas in a special theory T^{IQ} of FTT
- Ev_{oo} — (extension of) evaluative linguistic expression
- $\mu_{o(o\alpha)(o\alpha)}$ — measure

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"⟨Quantifier⟩ B's are A"

$$(Q^{\forall}_{Ev}\, x)(B, A) := \underbrace{(\exists z)[\,(\forall x)((B|z)\, x \Rightarrow Ax)}_{\text{Cut of } B \text{ giving the highest truth value}} \quad \wedge$$

$$\underbrace{Ev((\mu B)(B|z))]}_{\text{size of } B|z \text{ w.r.t. } B \text{ is evaluated by } Ev}$$

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Specific intermediate quantifiers

Theorem

A special case of intermediate quantifiers are the **classical** *ones:*

$$\textbf{A:}\ \textit{All B are A} := (Q^{\forall}_{Bi\boldsymbol{\Delta}}\, x)(B, A) \equiv (\forall x)(Bx \Rightarrow Ax),$$

$$\textbf{E:}\ \textit{No B are A} := (Q^{\forall}_{Bi\boldsymbol{\Delta}}\, x)(B, \neg A) \equiv (\forall x)(Bx \Rightarrow \neg Ax),$$

$$\textbf{I:}\ \textit{Some B are A} := (Q^{\exists}_{Bi\boldsymbol{\Delta}}\, x)(B, A) \equiv (\exists x)(Bx \wedge Ax),$$

$$\textbf{O:}\ \textit{Some B are not A} := (Q^{\exists}_{Bi\boldsymbol{\Delta}}\, x)(B, \neg A) \equiv (\exists x)(Bx \wedge \neg Ax).$$

Specific intermediate quantifiers

Non-classical quantifiers

P: Almost all B are $A := (Q^{\forall}_{Bi\,Ex}x)(B, A)$

$$(\exists z)[(\forall x)((B|z)x \Rightarrow Ax) \wedge \underbrace{(Bi\,Ex)}_{\text{Extremely big}} ((\mu B)(B|z))]$$

B: Almost all B are not $A := Q^{\forall}_{Bi\,Ex}(B, \neg A)$

T: Most B are $A := (Q^{\forall}_{Bi\,Ve}x)(B, A)$

D: Most B are not $A := Q^{\forall}_{Bi\,Ve}(B, \neg A)$

K: Many B are $A := (Q^{\forall}_{\neg(Sm\,\bar{\nu})}x)(B, A)$

G: Many B are not $A := (Q^{\forall}_{\neg(Sm\,\bar{\nu})}x)(B, \neg A)$

F: A few (A little) B's are $A := ((Q^{\forall}_{\neg\,Sm\,Ve}x)(B, A)$

V: A few (A little) B's are not $A := ((Q^{\forall}_{\neg\,Sm\,Ve}x)(B, \neg A)$

S: Several B's are $A := ((Q^{\forall}_{\neg\,Sm\,Si}x)(B, A)$

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Intermediate quantifiers semantically

$\mathscr{M}$ — a model
$B, A \subsetapprox M_\alpha$
μ — a measure on M_α
Ev — evaluative linguistic expression

Computation of intermediate quantifiers

$$\mathscr{M}(\text{Most } B\text{'s are } A) =$$
$$\bigvee \left\{ \bigwedge_{m \in M_\alpha} ((B|Z)(m) \to A(m)) \wedge \text{Very big}(\mu(B)(B|Z)) \,\middle|\, Z \subsetapprox M_\alpha \right\}$$

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Generalized Aristotle's syllogisms

Definition

- A syllogism $\langle \mathscr{P}_1, \mathscr{P}_2, \mathscr{C} \rangle$ — logical argument in which the *conclusion* $\mathscr{C}$ is inferred from two *premises*: *major* $\mathscr{P}_1$ and *minor* $\mathscr{P}_2$.
- A syllogism is valid if

$$T^{\mathrm{IQ}} \vdash \mathscr{P}_1 \,\&\, \mathscr{P}_2 \Rightarrow \mathscr{C}$$

By the completeness theorem:

$$\mathscr{M}(\mathscr{P}_1 \,\&\, \mathscr{P}_2 \Rightarrow \mathscr{C}) = 1$$

holds in every model of $\mathscr{M} \models T^{\mathrm{IQ}}$

$$\mathscr{M}(\mathscr{P}_1) \otimes \mathscr{M}(\mathscr{P}_2) \leq \mathscr{M}(\mathscr{C})$$

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Four figures of the generalized Aristotle's syllogisms

Figure I

$$\begin{array}{ll} \mathscr{P}_1: & Q_{\mathscr{P}_1}\ M \text{ are } P \\ \mathscr{P}_2: & Q_{\mathscr{P}_2}\ S \text{ are } M \\ \hline \mathscr{C}: & Q_{\mathscr{C}}\ S \text{ are } P \end{array}$$

Figure II

$$\begin{array}{ll} \mathscr{P}_1: & Q_{\mathscr{P}_1}\ P \text{ are } M \\ \mathscr{P}_2: & Q_{\mathscr{P}_2}\ S \text{ are } M \\ \hline \mathscr{C}: & Q_{\mathscr{C}}\ S \text{ are } P \end{array}$$

Figure III

$$\begin{array}{ll} \mathscr{P}_1: & Q_{\mathscr{P}_1}\ M \text{ are } P \\ \mathscr{P}_2: & Q_{\mathscr{P}_2}\ M \text{ are } S \\ \hline \mathscr{C}: & Q_{\mathscr{C}}\ S \text{ are } P \end{array}$$

Figure IV

$$\begin{array}{ll} \mathscr{P}_1: & Q_{\mathscr{P}_1}\ P \text{ are } M \\ \mathscr{P}_2: & Q_{\mathscr{P}_2}\ M \text{ are } S \\ \hline \mathscr{C}: & Q_{\mathscr{C}}\ S \text{ are } P \end{array}$$

$\mathscr{P}_1$ is major premise
$\mathscr{P}_2$ is minor premise
$\mathscr{C}$ is conclusion
$Q_{\mathscr{P}_1}, Q_{\mathscr{P}_2}, Q_{\mathscr{C}}$ are intermediate quantifiers

S is subject
P is predicate
M is middle formula

Valid generalized Aristotle's syllogisms

We proved that 105 out of over 4000 possible syllogisms are valid

Figure I	**Figure II**
$Q_{\mathscr{P}_1}$ M is P	$Q_{\mathscr{P}_1}$ P is M
$Q_{\mathscr{P}_2}$ S is M	$Q_{\mathscr{P}_2}$ S is M
$Q_{\mathscr{C}}$ S is P	$Q_{\mathscr{C}}$ S is P

Example (ATT-I)

All birds (M) are nicely singing (P)
Most animals in zoo (S) are birds (M)

Most animals in zoo (S) are nicely singing (P)

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$Q_{\mathscr{C}}$ S is P	$Q_{\mathscr{C}}$ S is P

Example (EKG-II)

No kings (P) are poor (M)
Many homeless people (S) are poor (M)

Many homeless people (S) are not kings (P)

Valid generalized Aristotle's syllogisms

Figure III

$$\frac{\begin{array}{l} Q_{\mathscr{P}_1}\ M \text{ is } P \\ Q_{\mathscr{P}_2}\ M \text{ is } S \end{array}}{Q_{\mathscr{C}}\ S \text{ is } P}$$

Figure IV

$$\frac{\begin{array}{l} Q_{\mathscr{P}_1}\ P \text{ is } M \\ Q_{\mathscr{P}_2}\ M \text{ is } S \end{array}}{Q_{\mathscr{C}}\ S \text{ is } P}$$

Example (PPI-III)

Almost all tall trees (M) are very thick (P)
Almost all tall trees (M) are prone to break (S)

Some trees prone to break (S) are very thick (P)

Valid generalized Aristotle's syllogisms

Figure III	Figure IV
$Q_{\mathscr{P}_1}$ M is P	$Q_{\mathscr{P}_1}$ P is M
$Q_{\mathscr{P}_2}$ M is S	$Q_{\mathscr{P}_2}$ M is S
$Q_{\mathscr{C}}$ S is P	$Q_{\mathscr{C}}$ S is P

Example (TAI-IV)

*Most shares with downward trend (P) are of car industry (M)
All shares of car industry (M) are important (S)

Some important shares (S) have downward trend (P)

Syllogisms semantically

Example (Algebraic form of a valid syllogism)

ATK-I :
$$\underbrace{\bigwedge_{u\in N}(M(u)\to P(u))}_{Q^{\forall}_{Bi\Delta}(M,P)}\otimes$$

$$\underbrace{\bigvee_{Z\in\mathscr{F}(N)}\left(\bigwedge_{u\in N}((S|Z)(u)\to M(u))\wedge Bi\,Ve(\mu(S,S|Z))\right)}_{Q^{\forall}_{Bi\,Ve}(S,M)}\leq$$

$$\underbrace{\bigvee_{Z\in\mathscr{F}(N)}\left(\bigwedge_{u\in N}((S|Z)(u)\to P(u))\wedge not\ Sm(\mu(S,S|Z))\right)}_{Q^{\forall}_{not\,Sm}(S,P)}$$

The algebraic core of syllogisms

Lemma

Let $\mathscr{E} = \langle E, \vee, \wedge, \otimes, \rightarrow, 0, 1\rangle$ *be MV-algebra. Then for all* $a, b, c \in E$

(a) $(b \rightarrow c) \otimes (a \rightarrow b) \leq a \rightarrow c$,

(b) $(b \rightarrow c) \otimes (a \wedge b) \leq a \wedge c$,

(c) $(a \rightarrow b) = (\neg b \rightarrow \neg a)$,

Theorem

All 105 valid syllogisms are the consequence of inequalities (a), (b) and equality (c) above.

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Generalized square of opposition

A scheme enabling us to derive new propositions from the known ones and to estimate their truth value

Subaltern-superaltern

- **Classical definition:** A formula P_2 is subaltern of P_1 (superaltern) if, in any model, it must be true if its superaltern is true.
  The superaltern P_1 must be false if the subaltern P_2 is false.
- **Fuzzy logic:** P_2 is subaltern of P_1 (and P_1 is superaltern of P_2) if

$$\vdash P_1 \Rightarrow P_2,$$
$$\mathscr{M}(P_1) \leq \mathscr{M}(P_2)$$

in any model $\mathscr{M}$.

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Contraries

- **Classical definition:** Formulas P_1, P_2 are contraries if in any model they cannot be both true but can be both false.
- **Fuzzy logic:**

$$\vdash P_1 \,\&\, P_2 \equiv \bot,$$
$$\mathscr{M}(P_1) \otimes \mathscr{M}(P_2) = 0$$

The strong conjunction $a \otimes b = \max\{0, a + b - 1\}$

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Subcontraries

- **Classical definition:** Formulas P_1, P_2 are subcontraries if in any model they cannot be both false but can be both true.
- **Fuzzy logic:** P_1, P_2 are subcontraries if

$$\vdash P_1 \,\boldsymbol{\nabla}\, P_2,$$

$$\mathscr{M}(P_1) \oplus \mathscr{M}(P_2) = 1$$

The strong disjunction $a \oplus b = \min\{1, a + b\}$

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Generalized square of opposition

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Contradictories

- **Classical definition:** Formulas P_1, P_2 are contradictories if in any model they neither can be both true, nor they can be both false.
- **Fuzzy logic:** $\vdash P_1 \mathbin{\&} P_2 \equiv \bot$ as well as $\vdash P_1 \nabla P_2$

$$\mathscr{M}(P_1) \otimes \mathscr{M}(P_2) = 0 \qquad \text{and} \qquad \mathscr{M}(P_1) \oplus \mathscr{M}(P_2) = 1$$

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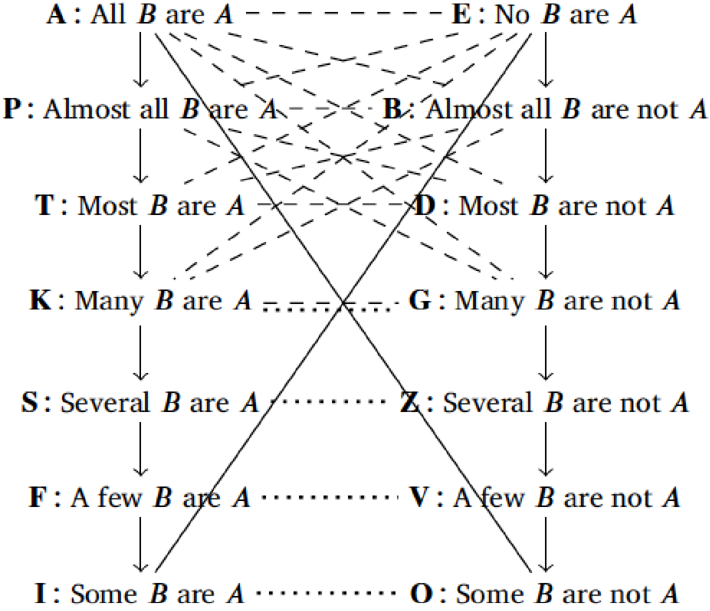
Graded square of opposition



Non-monotonic reasoning

- **Monotonic reasoning:** if $T \vdash A$, then $T' \vdash A$ for any $T' \supset T$.
  Adding information must not invalidate the previous conclusion.
- **Non-monotonic reasoning:** additional information may invalidate the previous conclusion (J. McCarthy, E. Sandewal, R. Reiter)
- Rules in commonsense reasoning have exceptions:
  university professors teach, birds fly, kids like ice-cream, Japanese cars are reliable, politicians lie

How to cope with exceptions?

Various kinds of non-monotonic logic (*default logic, preferential reasoning, defeasible logic, circumscription*)

Exceptions without contradiction are possible in Fuzzy Natural Logic

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Non-monotonic reasoning in fuzzy natural logic

J. McCarthy

Bird-penguin contradiction in classical monotonic logic

Let T be a first-order theory of classical predicate logic:

$$T \vdash (\forall x)(\mathrm{Bird}\,(x) \Rightarrow \mathrm{CanFly}\,(x)),$$
$$T \vdash (\exists x)(\mathrm{Bird}\,(x) \wedge \mathrm{Penguin}\,(x)),$$
$$T \vdash (\forall x)(\mathrm{Penguin}\,(x) \Rightarrow \neg\mathrm{CanFly}\,(x)).$$

Then $T \vdash (\exists x)(\mathrm{CanFly}\,(\mathbf{x}) \wedge \neg\mathrm{CanFly}\,(x))$ and so, T is contradictory.

Non-monotonic reasoning in fuzzy natural logic

Bird-penguin consistency in FNL

Let T be extension of T^{IQ} and Bird , Penguin , CanFly $\in \mathit{Form}_{o\alpha}$ be formulas. Let

$$T \vdash (\mathit{Most}\ x)(\mathrm{Bird}\, x, \mathrm{CanFly}\, x),$$
$$T \vdash (\exists x)\boldsymbol{\Delta}(\mathrm{Bird}\, x \wedge \mathrm{Penguin}\, x),$$
$$T \vdash (\forall x)(\mathrm{Penguin}\, x \Rightarrow \neg\mathrm{CanFly}\, x).$$

Then

$$T \vdash (\exists x)(\mathrm{Bird}\, x \wedge \neg\mathrm{CanFly}\, x)$$

and T is consistent, i.e., there is a model of T.

Sorites/falakros paradoxes are examples of non-monotonic reasoning

Falakros paradox

$$\text{Bald}\,(0), \text{Bald}\,(0) \Rightarrow \text{Bald}\,(1), \text{Bald}\,(1), \ldots,$$
$$\text{Bald}\,(n) \Rightarrow \text{Bald}\,(n+1), \text{Bald}\,(n+1), \ldots$$
$$(\forall n)(\text{Bald}\,(n) \Rightarrow \text{Bald}\,(n+1))$$

But we know that $\neg\text{Bald}\,(100000)$ (new information).

Then we prove

$$(\forall n)\text{Bald}\,(n) \wedge (\exists m)\neg\text{Bald}\,(m)$$ — **a contradiction**

The mistake:

$$\text{Bald}\,(n) \Rightarrow \text{Bald}\,(n+1)$$

cannot be fully true for all n

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Solution of Sorites/falakros paradoxes using intermediate quantifier

Theorem

Let T be a consistent extension of T^{IQ} containing Peano arithmetics. Given a formula $\mathbb{FN} \in Form_{o\sigma}$, σ a type for natural numbers. Let 0, $n, \mathbf{m}_0 \in Form_\sigma$. If

$$T \vdash \mathbb{FN}\, 0,$$
$$T \vdash (Not\ many\ n)((\mathbb{FN}\ n), (\mathbb{FN}(n+1) \vee (n \equiv \mathbf{m}_0)), \qquad ((*))$$
$$T \vdash (\exists n)\neg(\mathbb{FN}\, n)$$

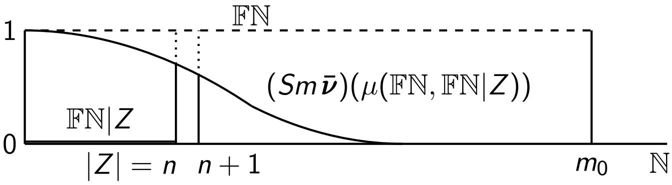
then T is consistent.

Scheme of sorites/falakros paradoxes using intermediate quantifier

Let $\mathbb{FN} = \{0, \dots, m_0\}$. The classical boolean implication

$$(\mathbb{FN}|Z)(n) \Rightarrow \mathbb{FN}(n+1) \vee n = m_0, \qquad n \in \mathbb{N}$$

is true for all $n \in N$, accompanied by a "measure of its credibility" for $|Z| = n$



$\mathbb{FN}(n) = 0$ for all $n > m_0$

Conclusion

We learned:

- The concept of Fuzzy Natural Logic
- The mathematical model of intermediate quantifiers, intermediate syllogisms and graded square of opposition
- FNL naturally resolves some problems of non-monotonic logic

6 graded Peterson's rules for checking validity of syllogisms

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Thank you for your attention!