Coalgebraic Dynamic Logic: Safety and Reducibility

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Joint work with Helle H. Hansen (RUG, Groningen)

Czech Gathering of Logicians & Beauty of Logic

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Motivation and Overview

Coalgebraic Modal Logic (CML)

- Modal logic for F-coalgebras
- Many-valued CML
  (cf. Kurz,P,Teheux 2023/24, Lin,Liau 2023, Bílková,Dostál 2013/16)
- Dynamic CML
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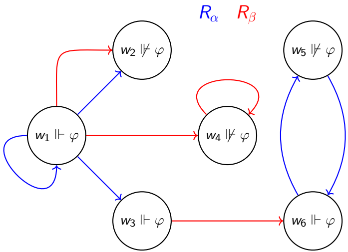
Aim: Combine many-valued and dynamic CML.

Overview of this talk:

- Concrete many-valued dynamic logics
- Many-valued dynamic CMLs
- Safe operations
- Reducible operations and strong completeness

Propositional Dynamic Logic (PDL)



$w_1 \Vdash \langle\alpha\rangle\varphi \quad w_1 \nVdash \langle\beta\rangle\varphi$

'It is possible that φ holds after executing program α'.
'It is not possible that φ holds after executing program β'.

[Fischer and Ladner, 1979]

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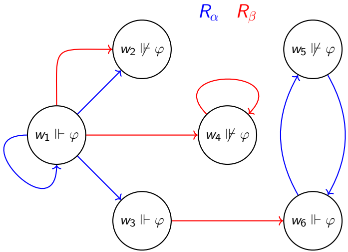

$w_1 \Vdash \langle\alpha\rangle\varphi \quad w_1 \not\Vdash \langle\beta\rangle\varphi$

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- $R_{\alpha\cup\beta} = R_\alpha \cup R_\beta$
- $R_{\alpha;\beta} = R_\alpha ; R_\beta = \{(w, v) \mid \exists u\colon wR_\alpha uR_\beta v\}$
- $R_{\alpha^*} = \bigcup_{n\geq 0} R_\alpha^n$ (reflexive transitive closure)
- $R_{\varphi?} = \{(w, w) \mid w \Vdash \varphi\}$

[Fischer and Ladner, 1979]

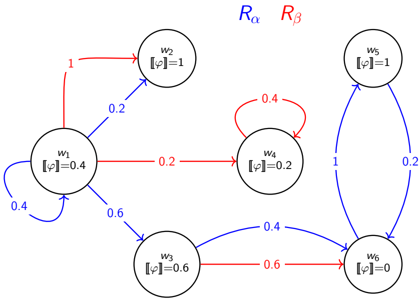
Beyond two

$$\mathbf{A} = \langle A, \wedge, \vee, \odot, \to, 0, 1 \rangle$$

Complete $\mathsf{FL}_{\mathrm{ew}}$-algebra
($\Leftrightarrow$ Complete commutative integral residuated lattice)
($\Leftrightarrow$ Commutative integral quantale)

Example. $[0, 1]$ with $\odot$ being the product of reals.

Many-valued PDL, labelled accessibility



$$[\![\langle\alpha\rangle\varphi]\!](w_1) = \bigvee R_\alpha(w_1, w_i) \odot [\![\varphi]\!](w_i) = 0.36$$

The maximal chance to reach φ after executing α is 0.36'.

Similarly to [Sedlàr, 2020] and further references therein

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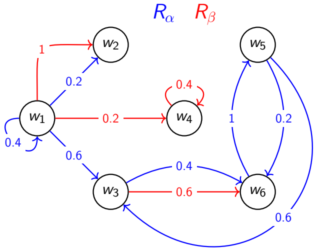
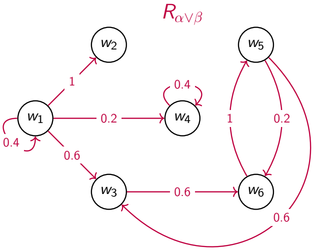
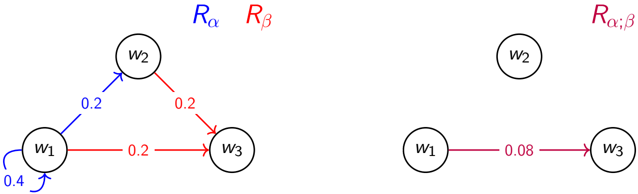




$$R_{\alpha\vee\beta} = R_\alpha \vee R_\beta$$

'*Maximal value to execute* α *or* β '

Many-valued PDL, labelled accessibility



$$R_{\alpha;\beta} = R_\alpha; R_\beta \qquad R_{\alpha;\beta}(x,z) = \bigvee\{R_\alpha(x,y) \odot R_\beta(y,z) \mid y \in X\}$$

'*Maximal value to execute first* α*, then* β '

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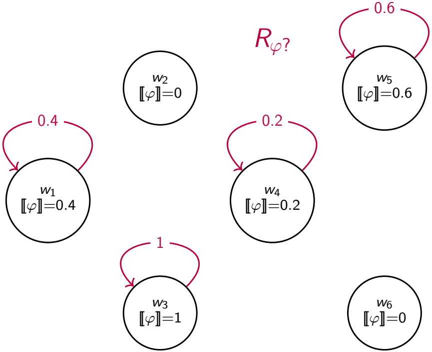
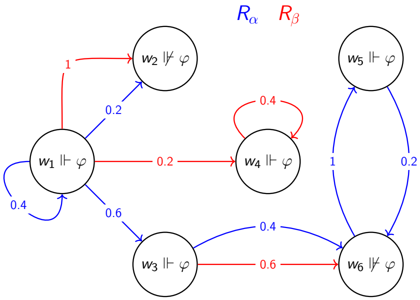


$$R_{\varphi?}(x,x) = [\![\varphi]\!](x) \qquad R_{\varphi?}(x,y) = 0 \text{ for } x \neq y$$

'*Test φ at state x* '

Two-valued PDL, labelled accessibility

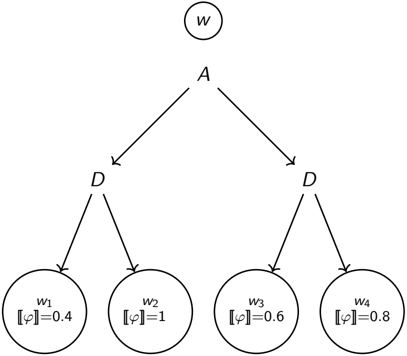


$$w_1 \Vdash \langle \alpha \rangle^{0.4} \varphi \qquad w_1 \nVdash \langle \beta \rangle^{0.6} \varphi$$

'The maximal chance to reach φ after executing α is ≥ 0.4'.
'All chances to reach φ after executing β are < 0.6'.

Many-valued Game Logic

Angel (A) and Demon (D) play the game α at state w.



$N_\alpha(w) = \{\{w_1, w_2\}, \{w_3, w_4\}\}\uparrow$ (Strategies of A)

$\nu_\alpha(w)(\llbracket\varphi\rrbracket) = \bigvee_{S \in N_\alpha(w)} \bigwedge_{z \in S} \llbracket\varphi\rrbracket(z) = 0.6$ (Best φ which A can ensure)

[Parikh, 1983] - Two-valued Game Logic

Game Logic

Game constructs:

$\alpha \cup \beta$	'*Angel chooses whether they play α or β*'
$\alpha;\beta$	'*First α is played, then β is played*'
α^*	'*Angel chooses how many (possibly zero) times α is played*'
α^∂	'*Game α with roles of Angel and Demon interchanged*'
φ?	'*If φ is false, Angel immediately loses*'

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- Mon. '**A**-nbhd' frame is $\gamma\colon X \to \mathcal{M}_{\mathbf{A}}(X) = \{\nu \in \mathbf{A}^{\mathbf{A}^X} \text{ mon.}\}$

Predicate Liftings

Transition systems are based on F-coalgebras $\gamma\colon X \to \mathsf{F}X$
Modalities correspond to **predicate liftings** $\lambda^{\heartsuit}_X\colon A^X \to A^{\mathsf{F}X}$

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Coalgebra Operations

Informal Definition (Coalgebra operation)

An n-ary *coalgebra operation* is an assignment

$$\vec{\gamma} = (\gamma_1, \ldots, \gamma_n) \colon X \to (\mathsf{F}X)^n \quad \mapsto \quad O(\vec{\gamma}) \colon X \to \mathsf{F}X.$$

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For example in PDL:

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(\gamma_\alpha, \gamma_\beta) \colon X \to (\mathcal{P}X)^2 &\quad \mapsto \quad \gamma_{\alpha \cup \beta} \colon X \to \mathcal{P}X \\
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$$\gamma_\alpha \colon X \to \mathcal{M}_{\mathbf{A}} \quad \mapsto \quad \gamma_{\alpha^\partial} \colon X \to \mathcal{M}_{\mathbf{A}}$$

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A collection of maps $\theta_X \colon (\mathsf{F}X)^n \to \mathsf{F}X$ *induces* coalgebra operation

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Generalises framework for Coalgebraic Dynamic Logic of [Hansen et al., 2014, Hansen and Kupke, 2015].

(Enjoy with or without monads)

Safety

A core feature of PDL [van Benthem, 1998]:

If $\mathfrak{M}$ and $\mathfrak{M}'$ are PDL models and $B \subseteq X \times X'$ is a bisimulation for all *atomic* programs, then B is a bisimulation for *all* programs.

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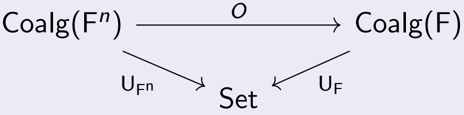
Smilarly for Game Logic [Pauly, 2000]:

If $\mathfrak{M}$ and $\mathfrak{M}'$ are game models and $B \subseteq X \times X'$ is a bisimulation for all *atomic* games, then B is a bisimulation for *all* games.

Safe Coalgebra Operation

Definition (Safe Coalgebra Operation)

A *safe coalgebra operation* is a functor $O\colon \mathrm{Coalg}(\mathsf{F}^n) \to \mathrm{Coalg}(\mathsf{F})$ for which the following commutes.



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$$\begin{array}{ccc} (\gamma_1, \gamma_2, \ldots, \gamma_n)\colon X \to (\mathsf{F}X)^n & \mapsto & O(\vec{\gamma})\colon X \to \mathsf{F}X \\ f \big\downarrow \;\; f \big\downarrow \;\; \cdots \;\; f \big\downarrow & \Longrightarrow & f \big\downarrow \\ (\gamma_1', \gamma_2' \ldots, \gamma_n')\colon Y \to (\mathsf{F}Y)^n & \mapsto & O(\vec{\gamma'})\colon Y \to \mathsf{F}Y \end{array}$$

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O^θ is safe if and only if $\theta \colon \mathsf{F}^n \Rightarrow \mathsf{F}$ is a natural transformation.

Examples (Sequential Composition and Iteration)

Collection of maps $\mu_X \colon \mathsf{FF}X \to \mathsf{F}X$ yields binary coalgebra operation $\gamma_1 * \gamma_2 \colon X \xrightarrow{\gamma_1} \mathsf{F}X \xrightarrow{\mathsf{F}\gamma_2} \mathsf{FF}X \xrightarrow{\mu_X} \mathsf{F}X$

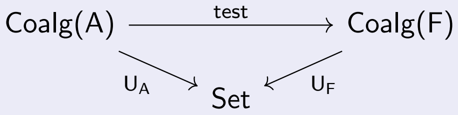
With $\eta_X \colon X \to \mathsf{F}X$ and $\bigsqcup_X \colon \mathcal{P}(\mathsf{F}X) \to \mathsf{F}X$ yields unary operation $\gamma^* = \bigsqcup_{k \in \omega} \gamma^{[k]}$ where $\gamma^{[0]} = \eta_X$ and $\gamma^{[k+1]} = \gamma^{[k]} * \gamma$

Composition $*$ is safe if $\mu \colon \mathsf{F} \circ \mathsf{F} \Rightarrow \mathsf{F}$ is a natural transformation. Iteration $(\cdot)^*$ is safe if $\eta \colon \mathrm{id} \Rightarrow \mathsf{F}$ and $\bigsqcup \colon \mathcal{P} \circ \mathsf{F} \Rightarrow \mathsf{F}$ are also natural.

Safe Tests and Naturality

Definition (Safe Test)

A *safe test* is a functor $\mathsf{test}\colon \mathsf{Coalg}(\mathsf{A}) \to \mathsf{Coalg}(\mathsf{F})$ for which the following commutes.



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$$\begin{array}{ccc} \mathsf{Coalg}(\mathsf{A}) & \xrightarrow{\mathsf{test}} & \mathsf{Coalg}(\mathsf{F}) \\ & {\scriptstyle \mathsf{U_A}} \searrow \quad \swarrow {\scriptstyle \mathsf{U_F}} & \\ & \mathsf{Set} & \end{array}$$

A test is safe if and only if it is a natural transformation $\mathsf{id} \Rightarrow \mathsf{F}(\text{-})^{A^{(\text{-})}}$.

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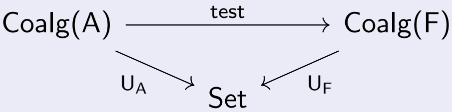

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Examples

All the tests discussed in this talk are safe.

Safety is nice

Proposition

Let all operations and tests be safe. If $\mathfrak{M}$ and $\mathfrak{M}'$ are coalgebraic dynamic models and $B \subseteq X \times X'$ is a bisimulation for all *atomic* actions, then B is a bisimulation for *all* actions.

Reducible Operations & Tests (Examples)

A-valued PDL:

$$\langle\alpha;\beta\rangle\varphi \leftrightarrow \langle\alpha\rangle\langle\beta\rangle\varphi, \quad \langle\alpha \vee \beta\rangle\varphi \leftrightarrow \langle\alpha\rangle\varphi \vee \langle\beta\rangle\varphi \quad \langle\psi?\rangle\varphi \leftrightarrow \psi \odot \varphi$$

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2-valued PDL, A-labelled accessibility (finite linear A):

$$\langle \alpha;\beta\rangle^{r}\varphi \leftrightarrow \bigvee_{r_1 \odot r_2 \geq r} \langle\alpha\rangle^{r_1}\langle\beta\rangle^{r_2}\varphi \quad \langle\psi?\rangle^{r}\varphi \leftrightarrow \psi \wedge \varphi$$

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A-valued Game Logic:

$$\langle \alpha^{\partial} \rangle \leftrightarrow \neg \langle \alpha \rangle \neg \varphi \quad \langle \psi ? \rangle \varphi \leftrightarrow \psi \odot \varphi$$

Reducibility is nice(r)

Proposition (Reducible $\Rightarrow$ Safe)

If collection of predicate liftings Λ is *separating*:

1. Every reducible coalgebra operation is safe.
2. Every reducible test is safe.

Axiomatizations

Let *every* $O \in \mathsf{Op}$ and test $\in$ Te be reducible.
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Definition. *Dynamic coalgebraic logic* $\mathcal{L}_\mathsf{F}^{\mathsf{Dyn}}$ is obtained as follows.

- Take $\mathcal{L}_\mathsf{F}$ instantiated at every $\alpha \in \mathsf{Act}$.
- Add all instances of reduction axioms for coalgebra operations.
- Add all instances of reduction axioms for tests.

Example: Iteration-free $\mathbf{A}$-valued Game Logic

For *finite* $\mathbf{A}$ combine:

- $\mathbf{A}$-valued propositional logic (*e.g.*, finite Łukasiewicz or Gödel)

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- Reduction axioms

$$\langle \alpha ; \beta \rangle p \leftrightarrow \langle \alpha \rangle \langle \beta \rangle p$$

$$\langle \alpha \vee \beta \rangle p \leftrightarrow \langle \alpha \rangle p \vee \langle \beta \rangle p$$

$$\langle \alpha \wedge \beta \rangle p \leftrightarrow \langle \alpha \rangle p \wedge \langle \beta \rangle p$$

$$\langle \alpha^{\partial} \rangle p \leftrightarrow \neg \langle \alpha \rangle \neg p$$

$$\langle \varphi ? \rangle p \leftrightarrow (\varphi \odot p)$$

Strong completeness (Sketch)

Canonical models: Replace maximally consistent sets with *non-modal homomorphisms* [Bou, et al., 2011]

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Theorem

For **A** finite, Λ separating, F *weakly preserving inverse limits of surjective ω-cochains*, $\mathcal{L}_{\mathsf{F}}$ *strongly one-step complete over finite sets*: There exists a canonical model on atomic actions.

Idea: Similarly to [Schröder and Pattinson, 2009], approach set of all formulas as inverse limit of finite sets of formulas, then apply one-step completeness of $\mathcal{L}_{\mathsf{F}}$.

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Presented a coalgebraic framework for many-valued dynamic logics.

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- Coalgebraic definitions of safety and reducibility.
- Coalgebraic strong completeness for iteration-free logics with reducible operations and tests.
  - *E.g.* new instances for iteration-free **A**-valued game logic, two-valued PDL with **A**-labelled accessibility.

Future Research

1. Aim for (weak) completeness including *non-reducible operations*, in particular *including iteration constructs*.
    - Reducibility $\rightsquigarrow$ General definition of *Fischer-Ladner closure*?
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Thanks!

More information found in the papers:
Conference version here. Extended version here.

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