Truthmaker Semantics and Curry-Howard Correspondence

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Outline

- truthmaker semantics for intuitionistic logic and its resemblance to BHK-interpretation
- typed lambda calculus as a formalization of BHK-interpretation
- productive truthmaker semantics for the typed lambda calculus

Truthmaker semantics

 

- van Fraassen (1969) Facts and tautological entailments. *The Journal of Philosophy*.
- Fine (2017) Truthmaker semantics. In: *A companion to the philosophy of language*.

State spaces

Truthmaker semantics is a modern version of situation semantics:

- in contrast to possible worlds states are partial;
- states form a mereological structure equipped with part-whole relation and a fusion operation;
- a state space is usually required to be a complete lattice

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Exact truthmaking

A state exactly verifies a proposition only if it is relevant as a whole to the truth of the proposition.

- a state exactly verifies a conjunction iff the state is obtained as fusion of two states that respectively exactly verify the conjuncts
- a state inexactly verifies a conjunction iff it inexactly verifies both conjuncts

applications: hyperintensionality, analytic implication, subject matter, counterfactuals, imperatives, scalar implicatures

Truthmaker sematnics for intuitionistic logic

Fine (2014). Truth-maker semantics for intuitionistic logic. *Journal of Philosophical Logic*.

- "a cross between construction-oriented semantics of Brouwer-Heyting-Kolmogorov and the condition-oriented semantics of Kripke"

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Truthmaker frames

A TM-frame is a complete Heyting algebra, i.e., a complete lattice in which finite meets distribute over arbitrary joins:

$$a \sqcap \bigsqcup_{i \in I} b_i = \bigsqcup_{i \in I} (a \sqcap b_i).$$

Relative pseudocomplement, the least element and the top element can be defined in the usual way:

$$a \Rightarrow b = \bigsqcup \{c \mid a \sqcap c \sqsubseteq b\}, \qquad 0 = \bigsqcup \varnothing, \qquad 1 = 0 \Rightarrow 0$$

The top element 1 can be called logical state.

Intended interpretation

- $a \sqcap b$ is the fusion of the states a and b
- $a \sqsubseteq b$ means that the state b is a part of the state a (b is included in a)

Truthmaker models

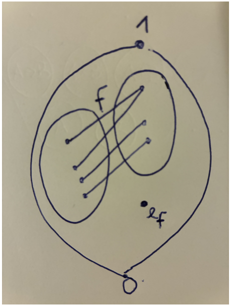
A TM-model is a tuple $\langle \mathcal{H}, V \rangle$, where $\mathcal{H}$ is a TM-frame and V is a function that assigns to every formula in $At \cup \{\bot\}$ a non-empty subset of H and satisfies the following falsum condition:

if $s \in V(\bot)$ then for all $\forall p \in At \; \exists t \in V(p) : s \sqsubseteq t$.

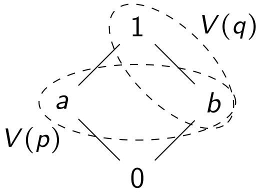
Representing functions by states

Let A, B be any non-empty subsets of H and $f : A \to B$. Then f can be encoded as a single element of H defined as follows:

$$e_f = \bigsqcap \{ a \Rightarrow f(a) \mid a \in A \}.$$



Example



	f_1	f_2	f_3	f_4
a	b	b	1	1
b	b	1	b	1
$e_{f_{\dots}}$	b	b	1	1

The codes of the functions are calculated as follows:

$$e_{f_1} = (a \Rightarrow b) \sqcap (b \Rightarrow b) = b \sqcap 1 = b;$$
$$e_{f_2} = (a \Rightarrow b) \sqcap (b \Rightarrow 1) = b \sqcap 1 = b;$$
$$e_{f_3} = (a \Rightarrow 1) \sqcap (b \Rightarrow b) = 1 \sqcap 1 = 1;$$
$$e_{f_4} = (a \Rightarrow 1) \sqcap (b \Rightarrow 1) = 1 \sqcap 1 = 1.$$

Exact truthmaking

- $a \Vdash p$ iff $a \in V(p)$, for every $p \in At \cup \{\bot\}$;
- $a \Vdash \varphi \rightarrow \psi$ iff there is a function f from the truthmakers of φ to the truthmakers of ψ such that $a = e_f$;
- $a \Vdash \varphi \wedge \psi$ iff there are b, c: $b \Vdash \varphi$, $c \Vdash \psi$ and $a = b \sqcap c$;
- $a \Vdash \varphi \vee \psi$ iff $a \Vdash \varphi$ or $a \Vdash \psi$.

Exact truthmaking

- $[p] = V(p)$, for every $p \in At \cup \{\bot\}$;
- $[\varphi \to \psi] = \{e_f \mid f : [\varphi] \to [\psi]\}$;
- $[\varphi \wedge \psi] = \{a \sqcap b \mid a \in [\varphi], b \in [\psi]\}$;
- $[\varphi \vee \psi] = [\varphi] \cup [\psi]$.

φ is valid in $\mathcal{M}$ if $1 \in [\varphi]$ in $\mathcal{M}$.

Completeness

Theorem (Fine, 2014)

$\vdash_{\mathsf{IL}} \varphi$ *if and only if* φ *is valid in every TM-model.*

BHK-interpretation of logical operators

- a proof of $\varphi \wedge \psi$ is given by two proofs: a proof of φ and a proof of ψ
- a proof of $\varphi \vee \psi$ is given by one proof: either by a proof of φ or by a proof of ψ
- a proof of $\varphi \to \psi$ is given by a construction that transforms every potential proof of φ into a proof of ψ
- a proof of $\neg\varphi$ is given by a construction that transforms every potential proof of φ into a contradiction

Propositional intuitionistic logic

$$\frac{\bot}{\varphi}\ (\text{EFQ})$$

$$\frac{\begin{array}{c}[\varphi]\\ \mathcal{D}\\ \psi\end{array}}{\varphi \to \psi}\ (\to\text{-intro}) \qquad \frac{\varphi \qquad \varphi \to \psi}{\psi}\ (\to\text{-elim})$$

$$\frac{\varphi \qquad \psi}{\varphi \wedge \psi}\ (\wedge\text{-intro}) \qquad \frac{\varphi \wedge \psi}{\varphi}\ (\wedge\text{-elim}) \quad \frac{\varphi \wedge \psi}{\psi}\ (\wedge\text{-elim})$$

$$\frac{\varphi}{\varphi \vee \psi}\ (\vee\text{-intro}) \quad \frac{\psi}{\varphi \vee \psi}\ (\vee\text{-intro}) \qquad \frac{\varphi \vee \psi \qquad \begin{array}{c}[\varphi]\\ \mathcal{D}_1\\ \chi\end{array} \qquad \begin{array}{c}[\psi]\\ \mathcal{D}_2\\ \chi\end{array}}{\chi}\ (\vee\text{-elim})$$

Terms in Lambda calculus

$$
\begin{aligned}
s, t, u := \; & x, y, z \mid efq_{\varphi}(t) \mid \lambda x.t \mid ap(s, t) \mid \langle s, t \rangle \mid fst(t) \mid snd(t) \mid \\
& \mid inl_{\varphi \vee \psi}(t) \mid inr_{\varphi \vee \psi}(t) \mid case(s, x.t, y.u)
\end{aligned}
$$

Typed lambda calculus

$$\frac{}{x^\varphi : \varphi}$$

$$\frac{t : \bot}{efq_\varphi(t) : \varphi}$$

$$\frac{t : \psi}{\lambda x^\varphi . t : \varphi \to \psi}$$

$$\frac{s : \varphi \qquad t : \varphi \to \psi}{ap(t, s) : \psi}$$

$$\frac{s : \varphi \qquad t : \psi}{\langle s, t \rangle : \varphi \wedge \psi}$$

$$\frac{t : \varphi \wedge \psi}{fst(t) : \varphi}$$

$$\frac{t : \varphi \wedge \psi}{snd(t) : \psi}$$

$$\frac{s : \varphi}{inl_{\varphi \vee \psi}(s) : \varphi \vee \psi}$$

$$\frac{s : \varphi \vee \psi \qquad t : \chi \qquad u : \chi}{case(s, x^\varphi . t, y^\psi . u) : \chi}$$

$$\frac{t : \psi}{inr_{\varphi \vee \psi}(t) : \varphi \vee \psi}$$

Closed lambda terms encode proofs

Assume $x : r \to p$, $y : r$, $z : p \to q$

From the expression

$$\lambda z \lambda x \lambda y.ap(z, ap(x, y)) : (p \to q) \to ((r \to p) \to (r \to q))$$

a proof can be reconstructed.

Open lambda terms encode derivations

Assume $x : (p \to r) \wedge (q \to r)$, $y : p \vee q$, $u : p$, $v : q$

From the expression

$$\lambda y.case(y, u.ap(fst(x), u), v.ap(snd(x), v)) : (p \vee q) \to r$$

a derivation can be reconstructed.

The equational theory of simply typed lambda calculus

$$t \equiv \langle fst(t), snd(t)\rangle \qquad fst(\langle s, t\rangle) \equiv s$$

$$snd(\langle s, t\rangle) \equiv t$$

$$t \equiv case(t, x.inl(x), y.inr(y)) \qquad case(inl(s), x.t, y.u) \equiv t[s/x]$$

$$case(inr(s), x.t, y.u) \equiv u[s/x]$$

$$t \equiv \lambda x.ap(t, x) \qquad ap(\lambda x.t, s) \equiv t[s/x]$$

$$\lambda x.t \equiv \lambda y.t[y/x] \qquad case(s, w.t, x.u) \equiv case(s, y.t[y/w], z.u[z/x])$$

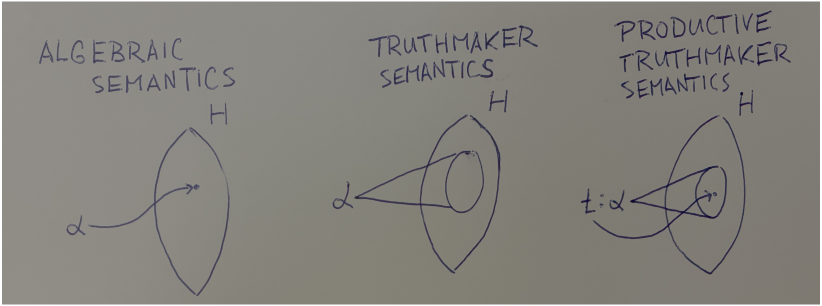
$$\frac{t \equiv s \qquad s \equiv u}{t \equiv u} \qquad \frac{s \equiv u}{t[u/x] \equiv t[s/x]}$$

Detours

$$\dfrac{\dfrac{\begin{matrix}[\varphi]^x \\ \mathcal{D}_1 \\ \psi\end{matrix}}{\varphi \to \psi}\,(\to\text{-intro}) \qquad \begin{matrix}\mathcal{D}_2 \\ \varphi\end{matrix}}{\psi}\,(\to\text{-elim}) \qquad \mapsto \qquad \begin{matrix}\mathcal{D}_2 \\ \varphi \\ \mathcal{D}_1 \\ \psi\end{matrix}$$

Computational rules and normalisation of proofs

$$
\dfrac{\dfrac{\begin{array}{c} x : \varphi \\ \mathcal{D}_1 \\ t : \psi \end{array}}{\lambda x.t : \varphi \to \psi}\ (\to\text{-intro}) \qquad \begin{array}{c} \mathcal{D}_2 \\ s : \varphi \end{array}}{ap(\lambda x.t, s) : \psi}\ (\to\text{-elim})
\quad \mapsto \quad
\begin{array}{c} \mathcal{D}_2 \\ s : \varphi \\ \mathcal{D}_1 \\ t[s/x] : \psi \end{array}
$$


ALGEBRAIC SEMANTICS
H
α
TRUTHMAKER SEMANTICS
H
α
PRODUCTIVE TRUTHMAKER SEMANTICS
H
t:α

A difference between truthmaker semantics and type theory

Klev, A. (2017). Truthmaker semantics: Fine versus Martin-Löf. In: *The Logica Yearbook 2016*.

Given different elements $a_1, a_2 \in [\varphi]$, $b_1, b_2 \in [\psi]$ such that $a_1 \sqcap b_1 = a_2 \sqcap b_2$, it is impossible to recover unambiguously a_1, b_1 from $a_1 \sqcap b_1$.

Interpreting formulas in a type theoretic hierarchy

Let us fix a TM-model $\mathcal{M} = \langle \mathcal{H}, V \rangle$. For every atomic formula p and every $a \in V(p)$ take a fresh copy c_a^p of a.

- $[\![p]\!] = \{c_a^p \mid a \in V(p)\}$, for every $p \in At \cup \{\bot\}$;
- $[\![\varphi \to \psi]\!] = [\![\varphi]\!] \to [\![\psi]\!]$ (the set of functions from $[\![\varphi]\!]$ to $[\![\psi]\!]$)
- $[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \times [\![\psi]\!]$ (Cartesian product of $[\![\varphi]\!]$ and $[\![\psi]\!]$)
- $[\![\varphi \vee \psi]\!] = [\![\varphi]\!] + [\![\psi]\!]$ (disjoint union of $[\![\varphi]\!]$ and $[\![\psi]\!]$)

Projecting the type theoretic hierarchy to the TM-model

For every $p \in At \cup \{\bot\}$

- $G(c_a^p) = a$

For every $f : [\![\varphi]\!] \to [\![\psi]\!]$:

- $G(f) = \bigsqcap\{G(k) \Rightarrow G(f(k)) \mid k \in [\![\varphi]\!]\}$

For every $k \in [\![\varphi]\!], l \in [\![\psi]\!]$:

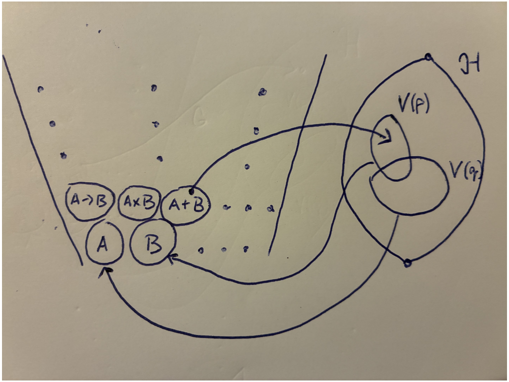
- $G(\langle k, l\rangle) = G(k) \sqcap G(l)$

For every $l \in [\![\varphi]\!]$:

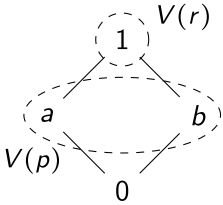
- $G(left_{\varphi\vee\psi}, l\rangle) = G(l)$

For every $k \in [\![\psi]\!]$:

- $G(\langle right_{\varphi\vee\psi}, k\rangle) = G(k)$

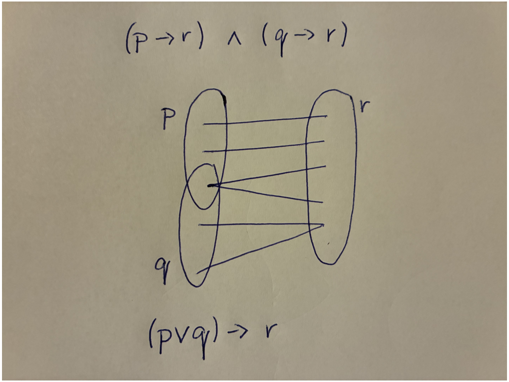

A→B
A×B
A+B
A
B
V(p)
V(q)
H

A deviation from the original framework



	g_1	g_2	g_3	g_4
$\langle left_{r\vee r}, 1\rangle$	c_a^p	c_a^p	c_b^p	c_b^p
$\langle right_{r\vee r}, 1\rangle$	c_a^p	c_b^p	c_a^p	c_b^p
$G(g_{\ldots})$	a	0	0	b

- $[(r \vee r) \to p] = \{a, b\}$;
- $G[\![(r \vee r) \to p]\!] = \{0, a, b\}$.


(p → r) ∧ (q → r)
p
r
q
(p∨q) → r

Completeness for intuitionistic logic

$$\downarrow X = \{a \in H \mid a \sqsubseteq b \text{ for some } b \in X\}.$$

Proposition

In every TM-model and for every L-formula φ*:*

$$\downarrow[\varphi] = \downarrow G[\![\varphi]\!].$$

As a consequence, $1 \in [\varphi]$ if and only if $1 \in G[\![\varphi]\!]$.

Theorem

$\vdash_{\mathsf{IL}} \varphi$ *if and only if* $1 \in G[\![\varphi]\!]$ *in every TM-model.*

Contradiction

Proposition

In every TM-model and for every formula φ it holds that $k \in [\![\bot]\!]$ only if there is $l \in [\![\varphi]\!]$ such that $G(k) \sqsubseteq G(l)$.

Hence in every TM-model we can fix a function $f^{\bot}_{\varphi} : [\![\bot]\!] \to [\![\varphi]\!]$ such that $G(m) \sqsubseteq G(f^{\bot}_{\varphi}(m))$, for every $m \in [\![\bot]\!]$.

Assignments

An assignment is a function i that assigns to every variable x^{φ} an element $i(x) \in [\![\varphi]\!]$.

An x-variant i^x_m of an assignment i is an assignment such that $i^x_m(x) = m$, and $i^x_m(y) = i(y)$ for any variable y different from x.

Interpreting terms in a type theoretic hierarchy

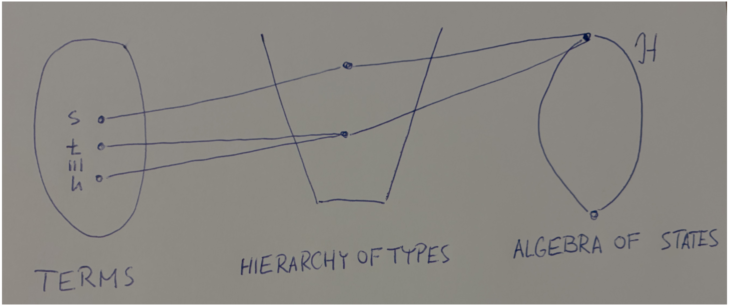
- $[\![x]\!]^i = i(x)$
- $[\![\lambda x^{\varphi}.t]\!]^i(m) = [\![t]\!]^{i^x_m}$, for every $m \in [\![\varphi]\!]$
- $[\![\langle t, s\rangle]\!]^i = \langle [\![t]\!]^i, [\![s]\!]^i \rangle$
- $[\![inl_{\varphi\vee\psi}(t)]\!]^i = \langle left_{\varphi\vee\psi}, [\![t]\!]^i \rangle$
- $[\![inr_{\varphi\vee\psi}(s)]\!]^i = \langle right_{\varphi\vee\psi}, [\![s]\!]^i \rangle$

Interpreting terms in a type theoretic hierarchy

- If $\vdash_{\lambda \mathrm{IL}} t : \varphi \to \psi$, $\vdash_{\lambda \mathrm{IL}} s : \varphi$ and $[\![t]\!]^i = f$ then $[\![ap(t,s)]\!]^i = f([\![s]\!]^i)$.
- If $\vdash_{\lambda \mathrm{IL}} t : \varphi \wedge \psi$ and $[\![t]\!]^i = \langle k, l \rangle$ then $[\![fst(t)]\!]^i = k$ and $[\![snd(t)]\!]^i = l$.
- If x, y are variables respectively of types φ, ψ and if $\vdash_{\lambda \mathrm{IL}} s : \varphi \vee \psi$, $\vdash_{\lambda \mathrm{IL}} t : \chi$, and $\vdash_{\lambda \mathrm{IL}} u : \chi$ then:

$$[\![case(s, x.t, y.u)]\!]^i = \begin{cases} [\![t]\!]^{i^x_k} & \text{if } [\![s]\!]^i = \langle 0, k \rangle, \text{ for some } k \in [\![\varphi]\!] \\ [\![u]\!]^{i^y_l} & \text{if } [\![s]\!]^i = \langle 1, l \rangle, \text{ for some } l \in [\![\psi]\!] \end{cases}$$

- If $t \in [\![\bot]\!]$ then $[\![efq_\varphi(t)]\!]^i = f^\bot_\varphi([\![t]\!]^i)$.


s
t
h
TERMS
HIERARCHY OF TYPES
ALGEBRA OF STATES

Proposition

Assume that t is a well-typed term. Then $\vdash_{\lambda\mathrm{IL}} t : \varphi$ if and only if in every TM-model and for every assignment i, $[\![t]\!]^i \in [\![\varphi]\!]$.

Proposition

Assume that t, s are well-typed terms and i is any assignment in any TM-model. Then:

(a) $G[\![\lambda x^{\varphi}.t]\!]^i = \bigsqcap_{k \in [\![\varphi]\!]} (G(k) \Rightarrow G[\![t]\!]^{i^x_k})$;

(b) $G[\![\langle t, s\rangle]\!]^i = G[\![t]\!]^i \sqcap G[\![s]\!]^i$;

(c) $G[\![inl(t)]\!]^i = G[\![inr(t)]\!]^i = G[\![t]\!]^i$.

Closure of a term

The closure of the term $t(x_1, \dots, x_n)$, denoted as $cl(t)$, is the closed term $\lambda x_1 \dots \lambda x_n . t$.

Proposition

Fix any TM-model. Let $x_1, \dots, x_n$ be variables respectively of types $\varphi_1, \dots, \varphi_n$, and let $t(x_1, \dots, x_n)$ be an open well-typed term of λIL. Then $[\![cl(t)]\!] = 1$ if and only if for every i and for every $k_1 \in [\![\varphi_1]\!], \dots, k_n \in [\![\varphi_n]\!]$:

$$G(k_1) \sqcap \dots \sqcap G(k_n) \sqsubseteq G[\![t]\!]^{i^{x_1}_{k_1}{}^{x_2}_{k_2}\dots{}^{x_n}_{k_n}}.$$

Proposition

Fix any TM-model and take any two well-typed terms s, t of λIL with the same free variables. Assume that for every i, $G[\![s]\!]^i \sqsubseteq G[\![t]\!]^i$. Then also $G[\![cl(s)]\!] \sqsubseteq G[\![cl(t)]\!]$.

Theorem

Assume that t is a well-typed closed term of λIL. Then $[\![t]\!]$ is a copy of the top element in every TM-model.

Corollary

Let $t(x_1, \ldots, x_n)$ be a well-typed term of λIL. Then for every i in any TM-model:

$$G[\![x_1]\!]^i \sqcap \ldots \sqcap G[\![x_n]\!]^i \sqsubseteq G[\![t]\!]^i.$$

Intuitively: $G[\![t]\!]^i$ is included in the fusion of $G[\![x_1]\!]^i, \ldots, G[\![x_n]\!]^i$

Corollary

$\vdash_{\mathrm{IL}} \varphi$ *if and only if there is a term t such that* $\vdash_{\lambda\mathrm{IL}} t : \varphi$ *and* $G[\![t]\!]^i = 1$ *in every TM-model and for every assignment i.*

Summary:

- a precisely defined connection between “condition-oriented” and “construction-oriented” exact semantics for intuitionistic logic was formulated
- a truthmaker semantics for typed lambda calculus was developed
- in this semantics derivations of statements are interpreted as truthmakers of these statements (types are interpreted by sets of states and terms as elements of these sets)

Future work:

- extension to richer type theories
- applications to logics of questions
- analytic implication/equivalence and Curry-Howard correspondence
- connections to domain theory