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Modal Weak Kleene Logics Through Variable Inclusion

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- Disclaimer: …—curiosity exceeds expertise
- *Modal* Weak Kleene Logics are relatively **unexplored/underdeveloped**
- Few Exceptions: Segerberg (1967), Correia (1998), Bonzio & Zamperlin (2024)
- Idea: approach Modal Weak Kleene logics *in the most classical way as possible*
- Heuristics & co.: exercise/exploration ⇒ motivation/foundation

Background

Language $\mathcal{L}$:

Classical prop. language $\mathcal{L}$ $(\wedge, \vee, \neg)$ over denumerable $Var = \{p, q, r, \dots\}$

Fm is the set of formulas of $\mathcal{L}$, $\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi$

$\varphi \to \psi := \neg\varphi \vee \psi$

Language $\mathcal{L}_\Box$:

$\mathcal{L}$ + unary $\Box$

$Fm_\Box$ is the set of formulas of $\mathcal{L}_\Box$, $\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Box\varphi$

$\Diamond\varphi := \neg\Box\neg\varphi$

- For $\Gamma \cup \{\varphi\} \subseteq Fm \cup Fm_\Box$,

  $\mathsf{var}(\varphi) = \{p \in Var \mid p \text{ occurs in } \varphi\}$

  $\mathsf{var}(\Gamma) = \bigcup_{\gamma \in \Gamma} \mathsf{var}(\gamma)$

Weak Kleene

We interpret $\mathcal{L}$

Weak Kleene Semantics

A weak Kleene valuation $[\![\cdot]\!] : Var \to \{0, 2, 1\}$ extends to all Fm according to the tables:

	$\neg$
1	0
2	2
0	1

$\wedge$	1	2	0
1	1	2	0
2	2	2	2
0	0	2	0

$\vee$	1	2	0
1	1	2	1
2	2	2	2
0	1	2	0

Idea:

$1(0) = [\![\varphi]\!] \approx \varphi$ is true(false)-and-on-topic

$2 = [\![\varphi]\!] \approx \varphi$ is off-topic

- $\vDash^{\mathbf{tt}} \subseteq \wp(Fm) \times Fm$:

$$\Gamma \vDash^{\mathbf{tt}} \varphi \quad \Leftrightarrow \quad \text{for all weak Kleene valuations } [\![\cdot]\!], \text{ if } [\![\gamma]\!] \in \{1,2\} \text{ for all } \gamma \in \Gamma, \text{ then } [\![\varphi]\!] \in \{1,2\}$$

- $\vDash^{\mathbf{ss}} \subseteq \wp(Fm) \times Fm$

$$\Gamma \vDash^{\mathbf{ss}} \varphi \quad \Leftrightarrow \quad \text{for all weak Kleene valuations } [\![\cdot]\!], \text{ if } [\![\gamma]\!] \in \{1\} \text{ for all } \gamma \in \Gamma, \text{ then } [\![\varphi]\!] \in \{1\}$$

Interpretation: $\Gamma \vDash^{\mathbf{ss}} \varphi$ iff φ is true whenever $\Gamma' s$ are true and the topic of φ is included in that of Γ

$\Gamma \vDash^{\mathbf{ss}} \varphi$ preserves truth and on-topic-ness

Infectious Lemma

For $\varphi \in Fm$, for all weak Kleene valuations $[\![\cdot]\!]$,

$$[\![\varphi]\!] = 2 \Leftrightarrow ([\![p]\!] = 2 \text{ for some } p \in \mathsf{var}(\varphi))$$

Idea: if a variable p is off-topic, φ containing p is off-topic too

Let $\vDash$ be classical logic consequence:

Theorem (Ciuni & Carrara, 2016)

$$\Gamma \vDash^{\mathbf{tt}} \varphi \Leftrightarrow \Gamma \vDash \varphi \text{ and OR } \begin{cases} \vDash \varphi & \\ \Gamma' \vDash \varphi & \begin{array}{l} \text{for some } \emptyset \neq \Gamma' \subseteq \Gamma \\ \text{such that } \mathsf{var}(\Gamma') \subseteq \mathsf{var}(\varphi) \end{array} \end{cases}$$

Theorem (Urquhart, 2001)

$$\Gamma \vDash^{\mathbf{ss}} \varphi \Leftrightarrow \Gamma \vDash \varphi \text{ and OR } \begin{cases} \Gamma \vDash p \wedge \neg p \\ \mathsf{var}(\varphi) \subseteq \mathsf{var}(\Gamma) \end{cases}$$

Idea: $\vDash^{\mathbf{ss}}$ is the topic-preserving fragment of classical logic:

$$\Gamma \vDash^{\mathbf{ss}} \varphi \Leftrightarrow \Gamma \vDash \varphi \text{ and OR } \begin{cases} \Gamma \vDash p \wedge \neg p \\ \text{the topic of } \varphi \text{ is included in that of } \Gamma \end{cases}$$

Goal: extend the above results to the <u>modal</u> weak Kleene setting

Modal Logic

Semantics for $\mathcal{L}_\Box$: Kripke Models

A Kripke model $\langle W, R, [\cdot]\rangle$ where:

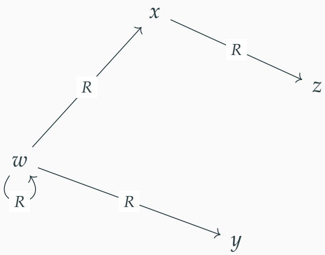
- $\langle W, R\rangle$ is a Kripke frame
  - $W \neq \emptyset$
  - $R \subseteq W \times W$

Notation: $xRy, R[x] = \{y \in W \mid xRy\}$

- $[\cdot] : Var \times W \to \{0,1\}$ is a valuation extended to all $Fm_\Box$ as follows:

$$
\begin{array}{lcl}
[\neg\varphi]_w = 1 & \Leftrightarrow & [\varphi]_w = 0 \\
[\varphi \wedge \psi]_w = 1 & \Leftrightarrow & [\varphi]_w = 1 \text{ and } [\psi]_w = 1 \\
[\varphi \vee \psi]_w = 1 & \Leftrightarrow & [\varphi]_w = 1 \text{ or } [\psi]_w = 1 \\
[\Box\varphi]_w = 1 & \Leftrightarrow & \text{for all } x \in W, \text{ if } wRx, \text{ then } [\varphi]_x = 1
\end{array}
$$

- We can describe the accessibility relations in n-steps: R^n
  - $wR^0x \Leftrightarrow w = x$ — w reaches itself with *no* step
  - $wR^{n+1}x \Leftrightarrow \exists y : wRy$ and yR^nx — if wRx, w reaches x in 1 step



wR^0w $\quad$ wR^1w

wR^1x

wR^1y

wR^2z

Semantics for $\mathcal{L}_\Box$: Kripke Models

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[\Box\varphi]_w = 1 & \Leftrightarrow & \text{for all } x \in W, \text{ if } wRx, \text{ then } [\varphi]_x = 1
\end{array}
$$

- Different **classes of Kripke frames** depending on the constraints on R

name	**condition**
K: all frames	none
D: seriality	$\forall x \exists y (xRy)$
...	...

- **Logical Consequence**: Let $\mathfrak{C}$ be a class of Kripke frames, $\models_{\mathfrak{C}} \subseteq \wp(Fm_\Box) \times Fm_\Box$:

$$\Gamma \models_{\mathfrak{C}} \varphi \quad \Leftrightarrow \quad \text{for all Kripke models } \langle W, R, [\cdot]\rangle \text{ whose frame is in } \mathfrak{C}, \text{ for all } w \in W \text{ if } [\gamma]_w = 1 \text{ for all } \gamma \in \Gamma, \text{ then } [\varphi]_w = 1$$

$$\Gamma \models_{\mathbf{D}} \varphi \quad \Leftrightarrow \quad \text{for all } \underline{\text{serial}} \text{ Kripke models } \langle W, R, [\cdot]\rangle, \text{ for all } w \in W \text{ if } [\gamma]_w = 1 \text{ for all } \gamma \in \Gamma, \text{ then } [\varphi]_w = 1$$

Modal Weak Kleene

Question: how to interpret $\mathcal{L}_\Box$ in a weak Kleene setting? Relativize values to worlds

1. Sub-question: how to interpret the Boolean fragment of $\mathcal{L}_\Box$ -formulas *with no* $\Box$?
   Sub-answer: worlds behave as weak Kleene valuations

Weak Kleene Semantics for $\mathcal{L}_\Box$

A Kripke-Kleene model $\langle W, R, [\![\cdot]\!] \rangle$ where:

- $\langle W, R \rangle$ is a Kripke frame
- $[\![\cdot]\!] : Var \times W \to \{0,1,2\}$ extends to all $Fm_\Box$ as follows:

$$\neg: \quad [\![\neg\varphi]\!]_w = \begin{cases} 1 \Leftrightarrow [\![\neg\varphi]\!]_w = 0 \\ 2 \Leftrightarrow [\![\neg\varphi]\!]_w = 2 \\ 0 \Leftrightarrow [\![\neg\varphi]\!]_w = 1 \end{cases}$$

$$\wedge: \quad [\![\varphi \wedge \psi]\!]_w = \begin{cases} 1 \Leftrightarrow [\![\varphi]\!]_w = 1 \text{ and } [\![\psi]\!]_w = 1 \\ 2 \Leftrightarrow [\![\varphi]\!]_w = 2 \text{ or } [\![\psi]\!]_w = 2 \\ 0 \Leftrightarrow \quad \text{otherwise} \end{cases}$$

$$\vee: \quad [\![\varphi \vee \psi]\!]_w = \begin{cases} 0 \Leftrightarrow [\![\varphi]\!]_w = 0 \text{ and } [\![\psi]\!]_w = 0 \\ 2 \Leftrightarrow [\![\varphi]\!]_w = 2 \text{ or } [\![\psi]\!]_w = 2 \\ 1 \Leftrightarrow \quad \text{otherwise} \end{cases}$$

$\Box$: ?

Question: how to interpret $\mathcal{L}_\Box$ in a weak Kleene setting? Relativize values to worlds

1. Sub-question: how to interpret the Boolean fragment of $\mathcal{L}_\Box$ -formulas *with no* $\Box$?
   Sub-answer: worlds behave as weak Kleene valuations Done ✓
2. Sub-question: how to interpret the Modal fragment of $\mathcal{L}_\Box$, -formulas *with* $\Box$?
   Sub-answer: let us start with the classical interpretation of $\Box\varphi$ in terms of quantifiers
   Translation Principle:

`It is necessary that` φ, `at` w	$\approx$	φ holds in all accessible worlds from w,
$\Box\varphi$ at w		$\forall x : wRx(\varphi \text{ at } x)$

Idea: let us look at weak Kleene semantics for first-order language

> *In general, the starting point consists in the so-called* ***substitutional conception*** *of quantifiers according to which* ∀ *and* ∃ *are treated as (infinite) generalizations of conjunction and disjunction, respectively.* *(Malinowski, 1998)*

- With $\langle W, R\rangle$ (domain+interpretation of R), for $[\![\cdot]\!]$ a (first-order) weak Kleene valuation:

$$1 = [\![\Box\varphi]\!]_w = [\![\forall x : wRx(\varphi \text{ at } x)]\!] = [\![(\varphi \text{ at } x_1) \wedge \cdots \wedge (\varphi \text{ at } x_n)]\!] \quad \Leftrightarrow \quad \text{for all } x : wRx, [\![\varphi]\!]_x = 1$$
$$2 = [\![\Box\varphi]\!]_w = [\![\forall x : wRx(\varphi \text{ at } x)]\!] = [\![(\varphi \text{ at } x_1) \wedge \cdots \wedge (\varphi \text{ at } x_n)]\!] \quad \Leftrightarrow \quad \text{for some } x : wRx, [\![\varphi]\!]_x = 2$$
$$0 = [\![\Box\varphi]\!]_w = [\![\forall x : wRx(\varphi \text{ at } x)]\!] = [\![(\varphi \text{ at } x_1) \wedge \cdots \wedge (\varphi \text{ at } x_n)]\!] \quad \Leftrightarrow \quad \text{otherwise}$$

Translation Principle/Substitutional Concept

$\Box(\Diamond)$ is treated as the corresponding weak Kleene $\forall$/conjunction ($\exists$/disjunction)

Goal: we obtained a full semantics for $\mathcal{L}_\Box$

1. Sub-question: how to interpret the Boolean fragment of $\mathcal{L}_\Box$ -formulas *with no* $\Box$?
   Done (like in standard weak Kleene) ✓
2. Sub-question: how to interpret the Modal fragment of $\mathcal{L}_\Box$, -formulas *with* $\Box$?
   Done (Translation Principle) ✓

Weak Kleene Semantics for $\mathcal{L}_\Box$

A Kripke-Kleene model $\langle W, R, [\![\cdot]\!]\rangle$ where:

- $\langle W, R\rangle$ is a Kripke frame
- $[\![\cdot]\!] : Var \times W \to \{0,1,2\}$ extends to all $Fm_\Box$ as follows:

$$\neg: \quad [\![\neg\varphi]\!]_w = \begin{cases} 1 \Leftrightarrow [\![\neg\varphi]\!]_w = 0 \\ 2 \Leftrightarrow [\![\neg\varphi]\!]_w = 2 \\ 0 \Leftrightarrow [\![\neg\varphi]\!]_w = 1 \end{cases}$$

$$\wedge: \quad [\![\varphi \wedge \psi]\!]_w = \begin{cases} 1 \Leftrightarrow [\![\varphi]\!]_w = 1 \text{ and } [\![\psi]\!]_w = 1 \\ 2 \Leftrightarrow [\![\varphi]\!]_w = 2 \text{ or } [\![\psi]\!]_w = 2 \\ 0 \Leftrightarrow \text{otherwise} \end{cases}$$

$$\vee: \quad [\![\varphi \vee \psi]\!]_w = \begin{cases} 0 \Leftrightarrow [\![\varphi]\!]_w = 0 \text{ and } [\![\psi]\!]_w = 0 \\ 2 \Leftrightarrow [\![\varphi]\!]_w = 2 \text{ or } [\![\psi]\!]_w = 2 \\ 1 \Leftrightarrow \text{otherwise} \end{cases}$$

$$\Box: \quad [\![\Box\varphi]\!]_w = \begin{cases} 1 \Leftrightarrow [\![\varphi]\!]_x = 1 \text{ for all } x : wRx \\ 2 \Leftrightarrow [\![\varphi]\!]_x = 2 \text{ for some } x : wRx \\ 0 \Leftrightarrow \text{otherwise} \end{cases}$$

Standard Translation (small digression)

Consider a first order language $\mathcal{L}^1_{FO}$ with:

- unary predicates $P_1, P_2, \ldots$, for each $p_1, p_2 \in Var$
- binary predicate R (accessibility relation)
- Generalized quantifiers: $\forall x : \Phi(x)$

Standard Translation: translates $\mathcal{L}_\Box$ into $\mathcal{L}^1_{FO}$

For every variable x in $\mathcal{L}_{FO}$, we can inductively define a translation function $\tau_x : Fm_\Box \to Fm_{FO}$ as follows:

$$\begin{array}{lcl} \tau_x(p_i) & = & P_i(x) \\ \tau_x(\neg\varphi) & = & \neg\tau_x(\varphi) \\ \tau_x(\varphi \wedge \psi) & = & \tau_x(\varphi) \wedge \tau_x(\psi) \\ \tau_x(\varphi \vee \psi) & = & \tau_x(\varphi) \vee \tau_x(\psi) \\ \tau_x(\Box\varphi) & = & \forall y : R(x,y)(\tau_y(\varphi)) \end{array}$$

where y is a fresh variable (i.e. a variable that has not been used earlier in the translation τ_x)

Kripke-Kleene

- $\langle W, R, [\![\cdot]\!] \rangle$

First-Order Weak Kleene

- $\langle W, R, [\![\cdot]\!] \rangle$ is a first-order structure where:
  - W is the domain of individuals
  - R interprets predicate R
  - $[\![\cdot]\!]$ interprets $P_1, P_2, \ldots$.

Standard Translation Theorem

For all variables x, for all models $\langle W, R, [\![\cdot]\!] \rangle$, for all $w \in W$,

$$[\![\varphi]\!]_w = [\![\tau_x(\varphi)[c_w/x]]\!]$$

Modal Weak Kleene Logics

- Different **classes of Kripke-Kleene models** depending on the constraints on R

name	**condition**
K: all models	`none`
D: seriality	$\forall x \exists y (xRy)$
...	...

- **Logical Consequences**: Let $\mathfrak{C}$ be a class of Kripke frames

$\vDash^{\mathbf{tt}}_{\mathfrak{C}} \subseteq \wp(Fm_{\Box}) \times Fm_{\Box}$:

$\Gamma \vDash^{\mathbf{tt}}_{\mathfrak{C}} \varphi \quad \Leftrightarrow \quad$ for all Kripke-Kleene models $\langle W, R, [\![\cdot]\!] \rangle$ whose frame is in $\mathfrak{C}$, for all $w \in W$ if $[\![\gamma]\!]_w \in \{1, 2\}$ for all $\gamma \in \Gamma$, then $[\![\varphi]\!]_w \in \{1, 2\}$

$\vDash^{\mathbf{ss}}_{\mathfrak{C}} \subseteq \wp(Fm_{\Box}) \times Fm_{\Box}$:

$\Gamma \vDash^{\mathbf{ss}}_{\mathfrak{C}} \varphi \quad \Leftrightarrow \quad$ for all Kripke-Kleene models $\langle W, R, [\![\cdot]\!] \rangle$ whose frame is in $\mathfrak{C}$, for all $w \in W$ if $[\![\gamma]\!]_w \in \{1\}$ for all $\gamma \in \Gamma$, then $[\![\varphi]\!]_w \in \{1\}$

Ex.

$\Gamma \vDash^{\mathbf{ss}}_{\mathbf{D}} \varphi \quad \Leftrightarrow \quad$ for all serial Kripke-Kleene models $\langle W, R, [\![\cdot]\!] \rangle$, for all $w \in W$ if $[\![\gamma]\!]_w \in \{1\}$ for all $\gamma \in \Gamma$, then $[\![\varphi]\!]_w \in \{1\}$

Idea: relationships with classical modal logics

Lemma

The following hold:

1. $\Gamma \vDash^{\mathbf{tt}}_{\mathfrak{C}} \varphi \Rightarrow \Gamma \vDash_{\mathfrak{C}} \varphi$
2. $\Gamma \vDash^{\mathbf{ss}}_{\mathfrak{C}} \varphi \Rightarrow \Gamma \vDash_{\mathfrak{C}} \varphi$
3. $\vDash^{\mathbf{tt}}_{\mathfrak{C}} \varphi \Leftrightarrow \vDash_{\mathfrak{C}} \varphi$
4. $\Gamma \vDash^{\mathbf{ss}}_{\mathfrak{C}} p \wedge \neg p \Leftrightarrow \Gamma \vDash_{\mathfrak{C}} p \wedge \neg p$

Characterizations

Recall:

Infectious Lemma

For $\varphi \in Fm$, for all weak Kleene valuations $[\![\cdot]\!]$,

$$[\![\varphi]\!] = 2 \Leftrightarrow ([\![p]\!] = 2 \text{ for some } p \in \mathsf{var}(\varphi))$$

Theorem (Ciuni & Carrara, 2016)

$$\Gamma \vDash^{\mathbf{tt}} \varphi \Leftrightarrow \Gamma \vDash \varphi \text{ and OR } \begin{cases} \vDash \varphi & \\ \Gamma' \vDash \varphi & \begin{array}{l}\text{for some } \emptyset \neq \Gamma' \subseteq \Gamma \\ \text{such that } \mathsf{var}(\Gamma') \subseteq \mathsf{var}(\varphi)\end{array} \end{cases}$$

Theorem (Urquhart, 2001)

$$\Gamma \vDash^{\mathbf{ss}} \varphi \Leftrightarrow \Gamma \vDash \varphi \text{ and OR } \begin{cases} \Gamma \vDash p \wedge \neg p \\ \mathsf{var}(\varphi) \subseteq \mathsf{var}(\Gamma) \end{cases}$$

Goal: extend the above results to the <u>modal</u> weak Kleene setting

Infectious Lemma

For $\varphi \in Fm$, for all weak Kleene valuations $[\![\cdot]\!]$,

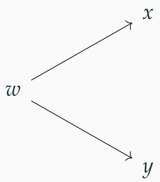
$$[\![\varphi]\!] = 2 \Leftrightarrow ([\![p]\!] = 2 \text{ for some } p \in \mathsf{var}(\varphi))$$

Question: in which sense the third value 2 is **infectious** in modal weak Kleene ?

$[\![p \vee q]\!]_w = 2$

$[\![p]\!]_w = 2$

$[\![p \wedge q]\!]_w = 2$



$[\![p]\!]_x = 1$

$[\![p]\!]_y = 1$

$[\![\Box p]\!]_w = 1$

- if φ doesn't contain $\Box$, then 2 propagates from sub-formulas to complex formulas
- p can take 2 at w without $\Box p$ taking 2.

- Idea: The presence of modal operators affects the propagation of the third value 2

The occurrence of a variable p in a formula $\varphi \in Fm_\Box$ under the scope of n-$\Box$'s:

Base: p occurs under the scope of 0-$\Box$'s in p;

$\neg, \wedge, \vee$: if p occurs under n-$\Box$'s in ψ, then, p occurs under the scope of n-$\Box$'s in $\neg\psi$, $\psi \vee \delta$, and $\psi \wedge \delta$;

$\Box$: if p occurs under the scope of n-$\Box$'s ψ, then it occurs under the scope of $n+1$-$\Box$'s in $\Box\psi$

Modal Deep Occurrence

For $\Gamma \cup \{\varphi\} \subseteq Fm_\Box$

- $\mathsf{var}^n(\varphi) = \{p \in Var \mid p$ occurs under the scope of n-modal operators $\Box$ in $\varphi\}$
- $\mathsf{var}^n(\Gamma) = \bigcup_{\varphi \in \Gamma} \mathsf{var}^n(\varphi)$

Ex.

- $\mathsf{var}^0(\Box(\neg p \vee \Box q)) = \emptyset$
- $\mathsf{var}^1(\Box(\neg p \vee \Box q)) = \{p\}$

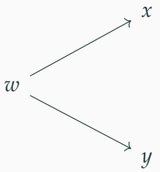
Modal Infectious Lemma

For $\varphi \in Fm_\Box$, for all Kripke-Kleene models $\langle W, R, [\![\cdot]\!]\rangle$ and all $w \in W$,

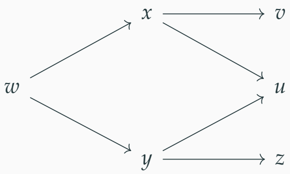
$$([\![\varphi]\!]_w = 2) \quad \Leftrightarrow \quad (\text{there are } n \in \mathbb{N}, x \in R^n[w], \text{ and } p \in \mathsf{var}^n(\varphi) \text{ such that } [\![p]\!]_x = 2)$$

$\mathsf{var}^{n>0} = \emptyset$ $\quad p \wedge q \quad$ w

$\mathsf{var}^{n=1} \neq \emptyset$ $\quad \Box p \wedge q$



$\mathsf{var}^{n=2} \neq \emptyset$ $\quad \Box(\Box p \wedge q)$



- The infectivity depends on the *n*-accessible worlds

Theorem

For $\Gamma \cup \{\varphi\} \subseteq Fm_\Box$,

$$\Gamma \vDash^{tt}_{\mathbf{D}} \varphi \Leftrightarrow \Gamma \vDash_{\mathbf{D}} \varphi \text{ and OR } \begin{cases} \vDash_{\mathbf{D}} \varphi & \\ \Gamma' \vDash_{\mathbf{D}} \varphi & \text{for some } \emptyset \neq \Gamma' \subseteq \Gamma \text{ such that} \\ & \text{for all } n \in \mathbb{N}, \mathsf{var}^n(\Gamma') \subseteq \mathsf{var}^n(\varphi) \end{cases}$$

- The $\vDash^{tt}_{\mathbf{D}}$ modal weak Kleene logic over serial Kripke-Kleene models corresponds to the left-to-right **modal sensitive companion** of the classical serial modal logic[1]

Theorem

For $\Gamma \cup \{\varphi\} \subseteq Fm_\Box$,

$$\Gamma \vDash^{ss}_{\mathbf{D}} \varphi \Leftrightarrow \Gamma \vDash_{\mathbf{D}} \varphi \text{ and OR } \begin{cases} \Gamma \vDash_{\mathbf{D}} p \wedge \neg p & \\ \mathsf{var}^n(\varphi) \subseteq \mathsf{var}^n(\Gamma) & \text{for all } n \in \mathbb{N} \end{cases}$$

- The $\vDash^{ss}_{\mathbf{D}}$ modal weak Kleene logic over serial Kripke-Kleene models corresponds to the right-to-left **modal sensitive companion** of the classical serial modal logic[2]

[1] When the conclusion is not valid.

[2] When the set of premises is satisfiable.

Discussion and Future Work

- Recall: results⇒motivation/foundation

Question: how can we interpret the above results? Do they have any application?

Topic Sensitivity

Modal Operators are not Topic Transparent

- $[\![\varphi]\!]_w = 2 \quad \approx \quad \varphi$ is off-topic at w
- $[\![\varphi]\!]_w = 1 \quad \approx \quad \varphi$ is true-and-on-topic at w
- $[\![\varphi]\!]_w = 0 \quad \approx \quad \varphi$ is false-and-on-topic at w
- truth/falsity and on/off-topicness are world-relative
  p can be off-topic 2 at w, while true and on topic at x
- p can be off-topic 2 at w while $\Box p$ is truth-and-topic at w
  because in all the accessible worlds $R[w]$, p may be true-and-on-topic
- what p is about at w may not be what p is about at x
- $\Box$ is not-topic-transparent: $\Box p$ may not be what p is about

John is human $\not\approx$ Necessarily, John is human

- $\models^{ss}_{\mathfrak{C}}$ is the logic preserving truth and on-topicness, under the assumption that modal operators $\Box$ are not topic transparent

Idea: Lewis's counterpart-theory interpretation of modal logic?

- Alert: the proof strategy of the above results relies on **point-generated submodels** and **unravelled models** applied to Kripke-Kleene models.

  convoluted and not modular

- Question: what about other modal logics, e.g. $\vDash^{\mathbf{tt}}_{\mathbf{T}}$ / $\vDash^{\mathbf{ss}}_{\mathbf{T}}$?

  Their Characterization is Open

  Ex. $\Box p \vDash^{\mathbf{ss}}_{\mathbf{T}} p$, $\mathsf{var}^{n}(p) \subseteq \mathsf{var}^{n+1}(\Box p)$

- Question: what about proof theory? (labelled sequent calculus?)

- Question: what about their global counterpart?

Thank You!

Language $\mathcal{L}_{FO}$: classical first-order language + restricted quantifiers: $\forall x : \Phi(x)$

First-Order Kleene

A Kleene first-order model is a structure $\langle D, [\![\cdot]\!]\rangle$ where:

- D is a non-empty set of individuals; constants $Con_D = \{c_d \mid d \in D\}$
- $At_D = \{P^n(c_1, \ldots, c_n) \mid c_i \in Con_D\}$
- $[\![\cdot]\!] : At_D \to \{0, 1, 2\}$ extended as follows:

$$\neg: \quad [\![\neg\Phi]\!] = \begin{cases} 1 \Leftrightarrow & [\![\Phi]\!] = 0 \\ 2 \Leftrightarrow & [\![\Phi]\!] = 2 \\ 0 \Leftrightarrow & \text{otherwise} \end{cases}$$

$$\wedge: \quad [\![\Phi \wedge \Psi]\!] = \begin{cases} 1 \Leftrightarrow & [\![\Phi]\!] = 1 \text{ and } [\![\Psi]\!] = 1 \\ 2 \Leftrightarrow & [\![\Phi]\!] = 2 \text{ or } [\![\Psi]\!] = 2 \\ 0 \Leftrightarrow & \text{otherwise} \end{cases}$$

$$\vee: \quad [\![\Phi \vee \Psi]\!] = \begin{cases} 0 \Leftrightarrow & [\![\Phi]\!] = 0 \text{ and } [\![\Psi]\!] = 0 \\ 2 \Leftrightarrow & [\![\Phi]\!] = 2 \text{ or } [\![\Psi]\!] = 2 \\ 1 \Leftrightarrow & \text{otherwise} \end{cases}$$

$$\forall: \quad [\![\forall x \Phi]\!] = \begin{cases} 1 \Leftrightarrow & \text{for all } d \in D, [\![\Phi(x/c_d)]\!] = 1 \\ 2 \Leftrightarrow & \text{for some } d \in D, [\![\Phi(x/c_d)]\!] = 2 \\ 0 \Leftrightarrow & \text{otherwise} \end{cases}$$

$$[\![\forall x : \Phi(x)(\Psi)]\!] = \begin{cases} 1 \Leftrightarrow & \text{for all } d \in D \text{ such that } [\![\Phi(x/c_d)]\!] = 1, [\![\Psi(x/c_d)]\!] = 1 \\ 2 \Leftrightarrow & \text{for some } d \in D \text{ such that } [\![\Phi(x/c_d)]\!] = 1, [\![\Phi(x/c_d)]\!] = 2 \\ 0 \Leftrightarrow & \text{otherwise} \end{cases}$$

Idea: we can *translate* our modal language into a first-order language $\mathcal{L}_{FO}$

Kripke-Kleene

- $\langle W, R, [\![\cdot]\!]\rangle$
- $\langle W, R\rangle$

First-Order Kleene

- $\langle D, [\![\cdot]\!]\rangle$
- $\langle W, R\rangle$ is a first-order structure where:
  - W is the domain of individuals
  - R is a (binary) predicate specifying which individual (world) is accessible by which individual

Standard Translation

For every variable x in $\mathcal{L}_{FO}$, we can inductively define a translation function $\tau_x : Fm_\Box \to Fm_{FO}$ as follows:

$$\begin{array}{lcl} \tau_x(p) & = & P(x) \\ \tau_x(\neg\varphi) & = & \neg\tau_x(\varphi) \\ \tau_x(\varphi \wedge \psi) & = & \tau_x(\varphi) \wedge \tau_x(\psi) \\ \tau_x(\varphi \vee \psi) & = & \tau_x(\varphi) \vee \tau_x(\psi) \\ \tau_x(\Box\varphi) & = & \forall y : R(x,y)(\tau_y(\varphi)) \end{array}$$

where y is a fresh variable (i.e. a variable that has not been used earlier in the translation τ_x)

Idea: $\tau_x(\varphi)$ translates in first-order language the "truth conditions" of φ at world x, i.e. the conditions under which φ holds at x

- Consider a Kripke-Kleene model $\mathcal{M} = \langle W, R, [\![\cdot]\!] \rangle$
- Consider its first-order counterpart $\mathcal{M}^{FO} = \langle W, [\![\cdot]\!] \rangle$ such that:
  - $Con_D = \{c_w \mid w \in W\}$
  - $[\![P(c_w)]\!] = 0/1/2 \Leftrightarrow [\![p]\!]_x = 0/1/2$
  - $[\![R(c_w, c_x)]\!] = 1 \Leftrightarrow wRx$, (0 otherwise)

Standard Translation Theorem

For all variables x,

$$[\![\varphi]\!]_w = [\![\tau_x(\varphi)[c_w/x]]\!]$$

In a Kripke-Kleene model $\mathcal{M} = \langle W, R, [\![\cdot]\!] \rangle$, the value taken by a formula φ at w corresponds to the value taken by its translation $\tau_x(\varphi)[c_w/x]$ in the first-order counterpart $\mathcal{M}^{FO} = \langle W, [\![\cdot]\!] \rangle$

Extends Standard Translation Theorem For Classical Modal Logic

Application: ...Conditionals

- (Variably Strict) Conditionals are restricted modal operators

if φ then ψ $\approx$ in all *relevant situations* where φ holds, ψ holds

$$\varphi \triangleright \psi \approx \Box_{\varphi}\psi$$

- $\langle W, R, [\![\cdot]\!]\rangle \quad \Rightarrow \quad \langle W, \{R_{\varphi}\}_{\varphi \in Fm_{\triangleright}}, [\![\cdot]\!]\rangle$ Multi-modal Kripke-Kleene models

- **Conditionals are not topic transparent**

- Conditional Weak Kleene Logics and Their Characterization:

  take into account the occurrence of a formula under n-conditional operators $\triangleright$