Knowledge on a Budget

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Topological space

$\langle X, \mathcal{T} \rangle$ a set X with $\mathcal{T} \subseteq \mathcal{P}(X)$, closed under finite intersections (i.e. $X \in \mathcal{T}$) and arbitrary unions (i.e. $\emptyset \in \mathcal{T}$).

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- **Topological evidence logic** [Baltag et al., 2016, 2022; Özgün, 2017] a hypothesis $P \subseteq X$ is justified iff $U \subseteq P$ for some dense open U ($\forall V \in \mathcal{T} : V \neq \emptyset \Rightarrow U \cap V \neq \emptyset$), or 'consistent with all consistent evidence'.

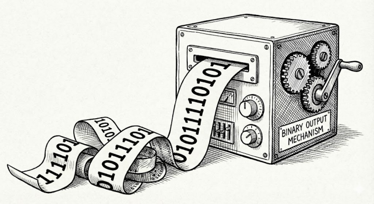
  P is known in x if justified by U and $x \in U$.

See also [Dabrowski et al., 1996; Parikh et al., 2007; Steinsvold, 2007].

Knowledge rarely comes for free.

Obtaining evidence requires resources and agents operate on strict resource budgets.

Example [Smyth, 1992; Vickers, 1989]



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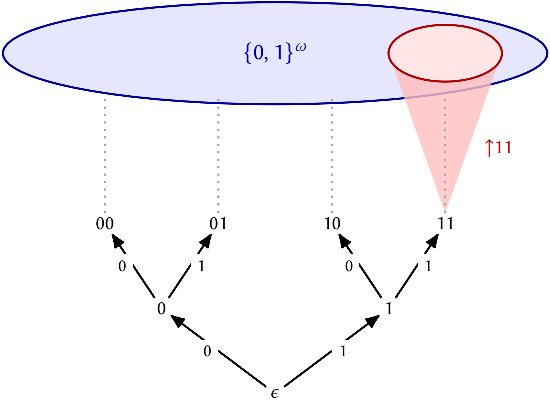

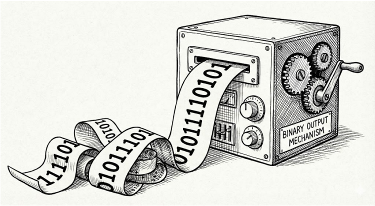

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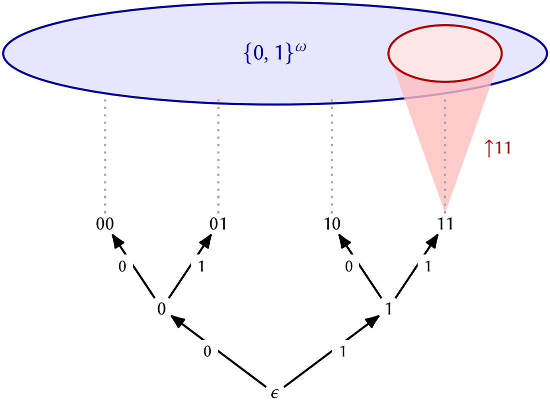

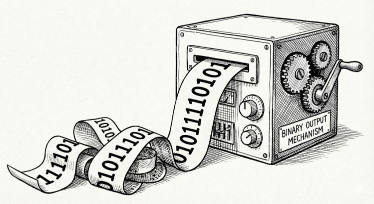

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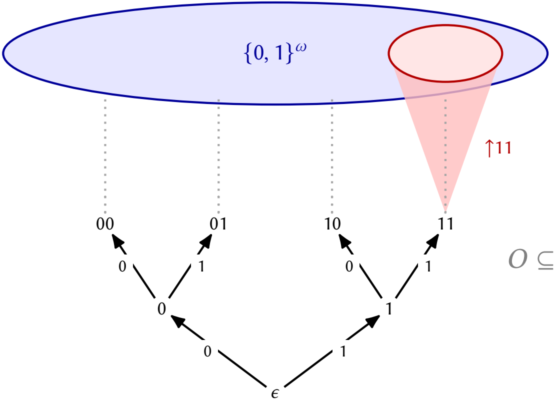

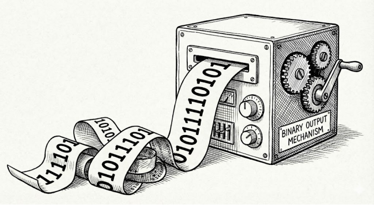

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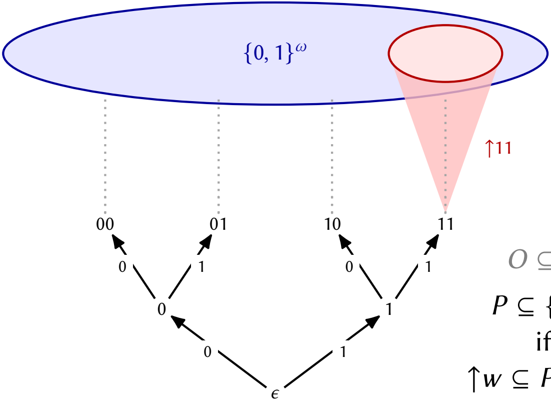

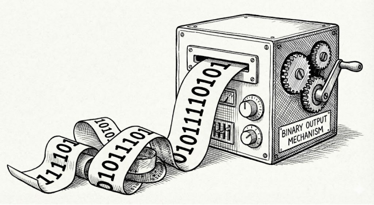

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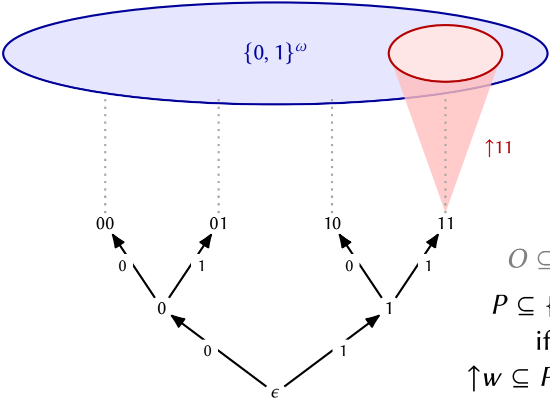

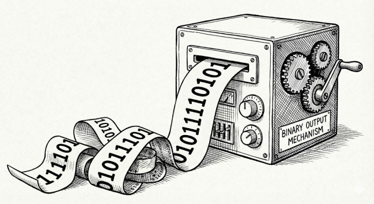

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P justified on budget n iff there is $w \in O$ where $\uparrow w \subseteq P$ & $|w| \leq n$
and $\forall u \in O : |u| \leq n \Rightarrow \uparrow u \cap P \neq \emptyset$.

Semiring

$\langle K, \oplus, \odot, \mathbb{0}, \mathbb{1} \rangle$:

$\langle K, \odot, \mathbb{1} \rangle$ monoid (resource combination)

$\langle K, \oplus, \mathbb{0} \rangle$ commutative monoid (resource choice)

$$a(b \oplus c) = ab \oplus ac \qquad (a \oplus b)c = ac \oplus bc \qquad \mathbb{0}a = \mathbb{0} = a\mathbb{0}$$

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Examples

- Tropical semiring $\langle \mathbb{N} \cup \{\infty\}, \min, +, \infty, 0\rangle$ discrete amounts of resources
- Viterbi semiring $\langle [0,1], \max, \cdot, 0, 1\rangle$ probabilities
- Łukasiewicz semiring $\langle [0,1], \max, \otimes, 0, 1\rangle$ truth degrees
- Binary relations $\langle \mathcal{P}(S \times S), \cup, \circ, \emptyset, \mathrm{id}_S\rangle$ actions / transitions
- ...

Semiring-annotated topological space (seat)

$\boldsymbol{S} = \langle X, \mathcal{T}, \mathcal{A}_K \rangle$ where $\langle X, \mathcal{T} \rangle$ is a topological space, $K = \langle K, \oplus, \odot, \mathbb{0}, \mathbb{1} \rangle$ is a semiring and

$$\mathcal{A}_K : \mathcal{T} \times X \longrightarrow \mathcal{P}(K)$$

$a \in \mathcal{A}(U, x)$ if resource a is sufficient to obtain evidence U in state x

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satisfies the following (for all $a \in K$, all $U, V \in \mathcal{T}$, and all $x \in X$):

$$a \in \mathcal{A}_K(U, x) \;\; \& \;\; b \in K \implies ab,\ ba \in \mathcal{A}_K(U, x) \tag{1}$$

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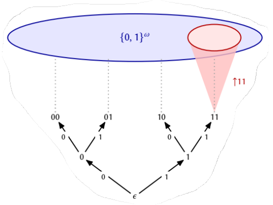
Def. $\mathcal{E}_a(x) := \{U \in \mathcal{T} \mid a \in \mathcal{A}_K(U, x)\}$ and $\mathcal{B}(x) := \{U \in \mathcal{T} \mid \mathcal{A}_K(U, x) \neq \varnothing\}$.



Example

- $X = \{0, 1\}^\infty$ binary words, finite + infinite
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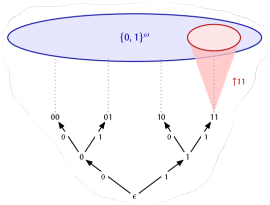

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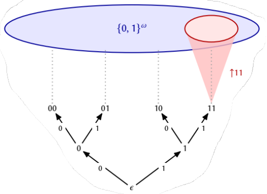

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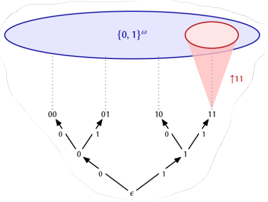

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- $a \in \mathcal{A}_K(U, w)$ iff $\exists u \in O(w) : \uparrow u \in U$ and $|u| \leq a$
  verifiable by a possible observation of length at most a

Special classes of seats:

- strong: $a \oplus b \in \mathcal{A}_x(U) \Rightarrow a, b \in \mathcal{A}_x(U)$ strong ideal
- bounded: $\mathbb{1} \in \mathcal{A}_x(X)$ and $\mathbb{0} \in \mathcal{A}_x(\emptyset)$ X for free, $\emptyset$ inaccessible
- uniform: $\mathcal{A}_x(U) = \mathcal{A}_y(U)$ availability not dependent on state
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Example (Cost seats from Borel measures): Let $\langle X, \mathcal{T}, \Sigma, \{\mu_x\}_{x \in X} \rangle$ where $\langle X, \Sigma \rangle$ is a measurable space and $\mathcal{T} \subseteq \Sigma$, and $\mu_x : X \longrightarrow \mathbb{R}_{\geq 0}$ is a measure for all $x \in X$.

- $K = \langle \mathbb{R}_{\geq 0} \cup \{\infty\}, \inf, +, \infty, 0 \rangle$
- $\mathcal{A}_K(U, x) = \{a \mid \mu_x(X \backslash U) \leq a\}$

E.g. probability measures (Markov transition systems): $a \in \mathcal{A}_x(U)$ is the probability of transitioning to $X \backslash U$ is not greater than a ('the acceptable risk').

Language $\mathfrak{L}_K$

Fix countable *Prop*. For each countable semiring K, define ($p \in Prop$, $a \in K$):

$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box\varphi \mid \mathrm{F}_a\varphi$$

$\Box\varphi$ 'there is factive evidence for φ' $\mathrm{F}_a\varphi$ 'there is evidence for φ available for $a \in K$'

$\Box_a\varphi := \Box\varphi \wedge \mathrm{F}_a\varphi$ 'there is factive evidence for φ available for a'

K-model

$\boldsymbol{M} = \langle X, \mathcal{T}, \mathcal{A}_K, \mathcal{V} \rangle$ where $\mathcal{V} : \mathit{Prop} \to \mathcal{P}(X)$ and

- $\boldsymbol{M}, x \models p$ iff $x \in \mathcal{V}(p)$
- $\boldsymbol{M}, x \models \neg\varphi$ iff $\boldsymbol{M}, x \not\models \varphi$
- $\boldsymbol{M}, x \models \varphi \wedge \psi$ iff $\boldsymbol{M}, x \models \varphi$ and $\boldsymbol{M}, x \models \psi$
- $\boldsymbol{M}, x \models \Box\varphi$ iff $\exists U\colon\ x \in U \in \mathcal{T}\ \ \&\ \ U \subseteq [\![\varphi]\!]_{\boldsymbol{M}}$ $\quad x \in \mathit{Int}([\![\varphi]\!]_{\boldsymbol{M}})$
- $\boldsymbol{M}, x \models \mathrm{F}_a\varphi$ iff $\exists U\colon\ a \in \mathcal{A}_K(U, x)\ \ \&\ \ U \subseteq [\![\varphi]\!]_{\boldsymbol{M}}$ $\quad x \in \mathit{For}_a([\![\varphi]\!]_{\boldsymbol{M}})$

where $[\![\varphi]\!]_{\boldsymbol{M}} = \{x \mid \boldsymbol{M}, x \models \varphi\}$. Local consequence $\models$ as usual.

Note: $\boldsymbol{M}, x \models \Box_a\varphi$ iff $a \in \mathcal{A}_x(\mathit{Int}([\![\varphi]\!]_{\boldsymbol{M}}))$.

Axiom system $S4_K$

S4 plus:

$$F_a\varphi \rightarrow (F_{ab}\varphi \wedge F_{ba}\varphi) \tag{6}$$

$$F_a\varphi \wedge F_b\psi \rightarrow F_{a\oplus b}(\Box\varphi \vee \Box\psi) \tag{7}$$

$$F_a\varphi \wedge F_b\psi \rightarrow F_{ab}(\varphi \wedge \psi) \tag{8}$$

$$F_{\mathbb{0}}\top \tag{9}$$

$$F_a\varphi \rightarrow F_a\Box\varphi \tag{10}$$

$$\frac{\varphi \rightarrow \psi}{F_a\varphi \rightarrow F_a\psi} \tag{11}$$

Derivability $\vdash_{\mathbf{S4}_K}$ defined as usual.

Theorem 1

$$\Gamma \vdash_{\mathbf{S4}_K} \varphi \quad \textit{iff} \quad \Gamma \models_{\mathbf{all}(K)} \varphi.$$

Axiom system S4sub$_K$

S4$_K$ plus:

$$F_{a\oplus b}\varphi \longrightarrow F_a\varphi \wedge F_b\varphi \tag{12}$$

$$F_{\mathbb{1}}\top \tag{13}$$

$$F_{\mathbb{0}}\bot \tag{14}$$

$$\pm F_a\varphi \longrightarrow F_{\mathbb{1}}\pm F_a\varphi \tag{15}$$

$$\pm F_a\varphi \longrightarrow \Box\pm F_a\varphi \tag{16}$$

Theorem 2

$$\Gamma \vdash_{\mathbf{S4sub}_K} \varphi \quad \textit{iff} \quad \Gamma \models_{\boldsymbol{sub}(K)} \varphi.$$

Proof of Theorem 2. The simple canonical model: X m.c. theories, $\mathcal{T}$ with basis $\{|\Box\varphi|\}_{\varphi \in \mathfrak{L}_K}$, $\mathcal{V}(p) = |p|$ and

$$a \in \mathcal{A}_\Gamma(U) \iff \exists\varphi : \mathrm{F}_a\varphi \in \Gamma \text{ and } |\Box\varphi| \subseteq U$$

However, F_a is a global modality ($\mathcal{A}(U)$ does not depend on the state), so the simple canonical model is not sufficient.

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However, F_a is a global modality ($\mathcal{A}(U)$ does not depend on the state), so the simple canonical model is not sufficient.

Pick a m.c. theory Λ; we *anchor* the simple canonical model in Λ:

- X^Λ = the set of $\Gamma \in X$ that agree with Λ on all $\mathrm{F}_a\varphi$
- $\mathcal{T}^\Lambda$ = subspace topology on X^Λ ($U \in \mathcal{T}^\Lambda$ iff $\exists V \in \mathcal{T}\colon U = V \cap X^\Lambda$)
- $A^\Lambda_\Gamma(U) = \bigcup\{\mathcal{A}_\Gamma(U) \mid V \cap X^\Lambda \subseteq U\}$
- $\mathcal{V}^\Lambda(p) = \mathcal{V}(p) \cap X^\Lambda$.

Then everything works: $\boldsymbol{M}^\Lambda$ is a s.u.b.-model and $|\chi|^\Lambda = [\![\chi]\!]_{\boldsymbol{M}^\Lambda}$.

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Fix countable *Prop*. For each countable semiring K, define ($p \in Prop$, $a \in K$):

$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box\varphi \mid \mathrm{F}_a\varphi \mid \mathrm{A}\varphi$$

$\mathrm{A}\varphi$ ‘φ holds in all states’ / ‘φ is assumed’

Recall from [Baltag et al., 2016, 2022]:

- $[\![\Diamond\varphi]\!] = X \backslash (Int(X \backslash [\![\varphi]\!])) = Cl([\![\varphi]\!])$

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- $[\![\Diamond\varphi]\!] = X \setminus (Int(X \setminus [\![\varphi]\!])) = Cl([\![\varphi]\!])$
- so, $[\![\mathrm{A}\Diamond\varphi]\!] = X$ iff $Cl([\![\varphi]\!]) = X$ iff $[\![\varphi]\!]$ is a *dense set* (if $U \in \mathcal{T}$ is non-empty, then $U \cap [\![\varphi]\!]$ is non-empty), otherwise it is empty

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What about φ *is justified on budget a?*

We can show, in uniform seats:

$$[\![\mathrm{A}\Diamond_a\Box_b\varphi]\!] = \begin{cases} X & \text{if } \exists U \in \mathcal{E}_b \colon U \subseteq [\![\varphi]\!] \text{ and } \forall V \in \mathcal{E}_a \colon V \neq \emptyset \Rightarrow V \cap U \neq \emptyset \\ \emptyset & \text{otherwise} \end{cases}$$

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Interesting properties:

- Superficial justification: $\mathrm{B}^\epsilon_\epsilon\varphi$ for some 'small' ϵ 'There is cheap evidence for φ that is consistent with all consistent cheap evidence'.

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- Susceptibility to disinformation: $\mathrm{B}^\epsilon_\epsilon\varphi \wedge \neg\Box\varphi$ 'φ can be justified superficially but there is no factive evidence for it'.
- Bias: $\mathrm{B}^\epsilon_a\varphi$ 'Some evidence for φ is easily accessible, but it also survives attacks of level a'.

Axiom system $\mathbf{S4sub}_K^{\mathrm{A}}$

$\mathbf{S4sub}_K$ plus

$$\mathrm{A}\varphi \longrightarrow \varphi \tag{17}$$

$$\mathrm{A}\varphi \longrightarrow \mathrm{AA}\varphi \tag{18}$$

$$\varphi \longrightarrow \mathrm{AE}\varphi \tag{19}$$

$$\mathrm{A}\varphi \wedge \mathrm{A}\psi \longrightarrow \mathrm{A}(\varphi \wedge \psi) \tag{20}$$

$$\mathrm{A}\varphi \longrightarrow \Box\varphi \wedge \mathrm{F}_{\mathbb{1}}\varphi \tag{21}$$

Theorem 3

$$\Gamma \vdash_{\mathbf{S4sub}^{\mathrm{A}}_{K}} \varphi \quad \textit{iff} \quad \Gamma \models_{\boldsymbol{sub}(K)} \varphi.$$

Summary:

- TEL + resources = a (more) realistic logic of evidence
- interesting concepts (e.g. disinformation)
- soundness and strong completeness
- and more... see arXiv (soon)

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To do:

- decidability and complexity
- multi-agent extensions
- dynamic extensions
- non-classical versions (e.g. relevant)

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S. Abramsky. Domain theory in logical form. *Annals of Pure and Applied Logic*, 51(1–2):1–77, Mar. 1991. ISSN 0168-0072. doi: 10.1016/0168-0072(91)90065-t.

A. Baltag, N. Bezhanishvili, A. Özgün, and S. Smets. Justified belief and the topology of evidence. In *Logic, Language, Information, and Computation*, pages 83–103. Springer Berlin Heidelberg, 2016. ISBN 9783662529218. doi: 10.1007/978-3-662-52921-8_6.

A. Baltag, N. Bezhanishvili, A. Özgün, and S. Smets. Justified belief, knowledge, and the topology of evidence. *Synthese*, 200(6), Dec. 2022. ISSN 1573-0964. doi: 10.1007/s11229-022-03967-6.

A. Dabrowski, L. S. Moss, and R. Parikh. Topological reasoning and the logic of knowledge. *Annals of Pure and Applied Logic*, 78(1–3):73–110, Apr. 1996. ISSN 0168-0072. doi: 10.1016/0168-0072(95)00016-x.

J. C. C. McKinsey and A. Tarski. The algebra of topology. *The Annals of Mathematics*, 45(1):141, Jan. 1944. ISSN 0003-486X. doi: 10.2307/1969080.

A. Özgün. *Evidence in Epistemic Logic: A Topological Perspective.* Phd thesis, ILLC, University of Amsterdam, 2017. URL `https://eprints.illc.uva.nl/id/eprint/2147/`. DS-2017-07.

R. Parikh, L. S. Moss, and C. Steinsvold. Topology and epistemic logic. In M. Aiello, I. Pratt-Hartmann, and J. van Benthem, editors, *Handbook of Spatial Logics*, pages 299–341. Springer Netherlands, 2007. ISBN 9781402055874. doi: 10.1007/978-1-4020-5587-4_6.

M. B. Smyth. Topology. In S. Abramsky, D. M. Gabbay, and T. S. E. Maibaum, editors, *Handbook of Logic in Computer Science*, volume 1, pages 641–761. Oxford University Press, Oxford, 1992. ISBN 9781383026023. doi: 10.1093/oso/9780198537359.003.0005.

C. Steinsvold. *Topological Models of Belief Logics.* Ph.D. Thesis, City University of New York, 2007.

S. Vickers. *Topology via Logic.* Cambridge University Press, 1989.